A CONSOLIDATED QUESTION PAPER-CUM-ANSWER BOOKLET

IMS™

(INSTITUTE OF MATHEMATICAL SCIENCES)

TEST SERIES (MAIN)-2014

Test Code: FULL TEST P-I(M) IAS / T- 11

MATHEMATICS

by K. VENKANNA

The person with 14 years of Teaching Experience

FULL TEST P-I

Time: Three Hours *Maximum Marks: 250*

INSTRUCTIONS

1. This question paper-cum-answer booklet has 52 pages and has 34 PART/SUBPART questions. Please ensure that the copy of the question paper-cum-answer booklet you have received contains all the questions.
2. Write your Name, Roll Number, Name of the Test Centre and Medium in the appropriate space provided on the right side.
3. A consolidated Question Paper-cum-Answer Booklet, having space below each part/sub part of a question shall be provided to them for writing the answers. Candidates shall be required to attempt answer to the part/sub-part of a question strictly within the pre-defined space. Any attempt outside the pre-defined space shall not be evaluated."
4. Answer must be written in the medium specified in the admission Certificate issued to you, which must be stated clearly on the right side. No marks will be given for the answers written in a medium other than that specified in the Admission Certificate.
5. Candidates should attempt Question Nos. 1 and 5, which are compulsory, and any THREE of the remaining questions selecting at least ONE question from each Section.
6. The number of marks carried by each question is indicated at the end of the question. Assume suitable data if considered necessary and indicate the same clearly.
7. Symbols/notations carry their usual meanings, unless otherwise indicated.
8. All questions carry equal marks.
9. All answers must be written in blue/black ink only. Sketch pen, pencil or ink of any other colour should not be used.
10. All rough work should be done in the space provided and scored out finally.
11. The candidate should respect the instructions given by the invigilator.
12. The question paper-cum-answer booklet must be returned in its entirety to the invigilator before leaving the examination hall. Do not remove any page from this booklet.

READ INSTRUCTIONS ON THE LEFT SIDE OF THIS PAGE CAREFULLY

Name: Nitish.K

Roll No.: 149709

Test Centre: Bangalore

Medium: English

Do not write your Roll Number or Name anywhere else in this Question Paper-cum-Answer Booklet.

I have read all the instructions and shall abide by them

Nitish.K

Signature of the Candidate

I have verified the information filled by the candidate above

Signature of the invigilator

IMPORTANT NOTE:
Whenever a question is being attempted, all its parts/ sub-parts must be attempted contiguously. This means that before moving on to the next question to be attempted, candidates must finish attempting all parts/ sub-parts of the previous question attempted. This is to be strictly followed. Pages left blank in the answer-book are to be clearly struck out in ink. Any answers that follow pages left blank may not be given credit.

P.T.O.

INDEX TABLE

QUESTION	No.	PAGE NO.	MAX. MARKS	MARKS OBTAINED
1	(a)			
	(b)			
	(c)			
	(d)			
	(e)			
2	(a)			
	(b)			
	(c)			
	(d)			
3	(a)			
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4	(a)			
	(b)			
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	(b)			
	(c)			
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			Total Marks	

SECTION-A

1. (a) Let R^+ be the set of all positive real numbers. Define the operations of addition and scalar multiplication as follows:

$$u+v=u.v \text{ for all } u,v\in R^+$$
$$\alpha u=u^{\alpha} \text{ for all } u\in R^+$$

and real scalar α. prove that R^+ is a real vector space. **(10)**

① $u+v = uv$ & $v+u = vu = uv$

as u & $v \in R^+$

$\Rightarrow u+v = v+u$.

② $u+1 = u\cdot 1 = u = 1+u$.

$u+1 = u = 1+u \Rightarrow 1$ is the additive inverse.

③ $u+u^{-1} = uu^{-1} = 1$

$u^{-1}+u = u^{-1}u = 1$.

$\therefore u^{-1}$ is multiplicative inverse.

④ $(u+v)+w = uv+w = uvw$

$u+(v+w) = u+vw = uvw$

$\therefore (u+v)+w = u+(v+w)$

⑥ $(\alpha+\beta)u$

$= u^{\alpha+\beta} = u^{\alpha}u^{\beta}$

$\alpha u+\beta u$

$= u^{\alpha}+u^{\beta}$

$= u^{\alpha}\cdot u^{\beta}$

$\Rightarrow (\alpha+\beta)u$

$= \alpha u+\beta u$.

⑤ $\alpha(u+v) = \alpha(uv) = (uv)^{\alpha} = u^{\alpha}v^{\alpha}$.

$\alpha u+\alpha v = (\alpha u)(\alpha v) = u^{\alpha}\cdot v^{\alpha}$

$\therefore \alpha(u+v) = \alpha u+\alpha v$.

$\therefore R^+$ is a real vector space.

1\. (b) Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by $T(x_1, x_2, x_3) = (x_1 + 3x_2 + 2x_3, 3x_1 + 4x_2 + x_3, 2x_1 + x_2 - x_3)$. Then find the dimension of the range space of T^2. Also find the dimension of the null space of T^3. (10)

$$T = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 4 & 1 \\ 2 & 1 & -1 \end{bmatrix} ; \quad T^2 = \begin{bmatrix} 14 & 17 & 3 \\ 17 & 26 & 9 \\ 3 & 9 & 6 \end{bmatrix}$$

$$T^3 = \begin{bmatrix} 71 & 113 & 42 \\ 113 & 164 & 51 \\ 42 & 51 & 9 \end{bmatrix}$$

$\text{Im}(T^2) = \text{col space}(T^2)$

$$T^2 \sim \begin{bmatrix} 14 & 17 & 3 \\ 0 & 75/14 & 75/14 \\ 0 & 75/14 & 75/14 \end{bmatrix} \sim \begin{bmatrix} 14 & 17 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ only two linearly independent columns

$\Rightarrow \dim(\text{col space } T^2) = 2$

$\Rightarrow$ $\underline{\text{Im}(T^2) \text{ has dimension } 2.}$

$T^3(x) = 0.$

$$\begin{bmatrix} 71 & 113 & 42 \\ 0 & -\frac{1125}{71} & -\frac{1125}{71} \\ 0 & -\frac{1125}{71} & -\frac{1125}{71} \end{bmatrix} \sim \begin{bmatrix} 71 & 113 & 42 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore \dim(\ker T^3) = n - r = 3 - 2 = 1.$

$\dim(\text{Null space}(T^3)) = 1.$

1. (c) Show that the following function is discontinuous at (0, 0): $f(x,y) = \begin{cases} \dfrac{x^3 + y^3}{x - y}, & x \neq y, \\ 0, & x = y. \end{cases}$

(10)

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} f(x, x - mx^3)$$

$$= \lim_{x\to 0} \frac{x^3 + (x - mx^3)^3}{mx^3}$$

$$= \lim_{x\to 0} \frac{1 + (1 - mx^2)^3}{m} = \frac{2}{m}.$$

$\therefore$ limit is different for different values of m.

$\therefore \lim_{(x,y)\to(0,0)} f(x,y)$ doesnot exist

$\therefore f(x,y)$ is discontinuous at $(0,0)$

1. (d) Find the equations of the line through the point $(-4, 3, 1)$ parallel to the plane $x+2y-z=5$ so as to intersect the line $-\frac{1}{3}(x+1)=\frac{1}{2}(y-3)=-(z-2)$. Find also the point of intersection. (10)

$$\frac{x+4}{l} = \frac{y-3}{m} = \frac{z-1}{n} \quad \text{— (1)}.$$

where $l+2m-n=0$. — (2)

as (1) and given line intersects.

$$\begin{vmatrix} +3 & 0 & 1 \\ -3 & 2 & -1 \\ l & m & n \end{vmatrix} = 0$$

P.T.O.

$\Rightarrow 3(2n+m)+(-3m-2l)=0 \Rightarrow \boxed{l=3n}$

Putting in ② $\Rightarrow m=-n$

$\therefore \frac{l}{3}=\frac{m}{-1}=\frac{n}{1}$

$\therefore$ eqn of the line $\boxed{\frac{x+4}{3}=\frac{y-3}{-1}=\frac{z-1}{1}}=r_1$ (say)

Any point $(-4+3r_1, 3-r_1, 1+r_1)$. on ①.

Any point on given line $(-1-3r_2, 3+2r_2, 2-r_2)$

equating $\Rightarrow -4+3r_1=-1-3r_2 \Rightarrow \underline{3r_1+3r_2=3}$.

$3-r_1=3+2r_2 \Rightarrow \underline{r_1+2r_2=0} \Rightarrow r_1=2, r_2=-1$

$\therefore$ point of intersection is $\underline{\underline{(2,1,3)}}$.

1. (e) Spheres are described to contain the circle $z=0., x^2+y^2=a^2$. Prove that the locus of the extremities of their diameters which are parallel to the x-axis is the rectangular hyperbola $x^2-z^2=a^2, y=0$ **(10)**

Given circle $x^2+y^2+z^2=a^2;\ z=0.$ — ①

eqn of sphere through ①

$\underline{x^2+y^2+z^2-a^2+\lambda z=0.}$

centre is $(0,0,-\lambda/2)$

eqn of any diameter is

$\frac{x-0}{1}=\frac{y-0}{0}=\frac{z+\lambda/2}{0}=r$ (say)

Any point on diameter $\Rightarrow (r, 0, -\lambda/2)$. — ②

if This is the extremity (x_1, y_1, z_1) then

$x_1 = r \; ; \; y_1 = 0 ; \; z_1 = -\lambda/2 .$

The extremity given by ② lies on given sphere

$$r^2 + 0 + \frac{\lambda^2}{4} - a^2 + \lambda\left(-\frac{\lambda}{2}\right) = 0$$

$$\Rightarrow r^2 - a^2 = \frac{\lambda^2}{4}$$

Put $r = x_1 , \quad \frac{\lambda}{2} = -z_1$

$$\Rightarrow x_1^2 - a^2 = z_1^2 \; ; \; y_1 = 0 .$$

$\therefore$ locus of extremity (x_1, y_1, z_1) is

$$x^2 - a^2 = z^2 \;\; \& \;\; y = 0$$

$$\Rightarrow \boxed{x^2 - z^2 = a^2 ; \;\; y = 0}$$

2. (a) (i) If $A = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix}$, then find A^{50}.

(ii) Let T: $P_3 \rightarrow P_3$ be the map given by $T(P(x)) = \int_1^x P'(t)\,dt$. If matrix of T relative to the standard basis $B_1 = B_2 = \{1, x, x^2, x^3\}$ is M and M^T denotes the transpose of the matrix M, then find $M+M^T$.

(19)

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}; \qquad \Rightarrow |A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & 0 \\ 1 & 0-\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(\lambda^2 - 1) = 0 \quad \Rightarrow \underline{\lambda = 1, 1, -1}.$$

By cayley hamilton theorem

$$A^3 - A^2 - A + I = 0 \Rightarrow \underline{A^3 = A^2 + A - I}.$$

But using above identity repeatedly we get

$$A^{50} = \begin{bmatrix} 1 & 0 & 0 \\ 25 & 1 & 0 \\ 25 & 0 & 1 \end{bmatrix}$$

(ii) $T(1) = \int_1^x 0\, dt = 0$

$T(x) = \int_1^x 1\, dt = x - 1 = -1 + x$

$T(x^2) = \int_1^x 2t\, dt = x^2 - 1 = -1 + 0x + 1x^2$

$T(x^3) = \int_1^x 3t^2\, dt = x^3 - 1 = -1 + 0x + 0x^2 + 1 \cdot x^3$

$$M = [T] = \begin{bmatrix} 0 & -1 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$M^T = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \Rightarrow M + M^T = \begin{bmatrix} 0 & -1 & -1 & -1 \\ -1 & 2 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ -1 & 0 & 0 & 2 \end{bmatrix}$$

P.T.O.

2. (b) (i) Find the values of a, b, and c, so that

$$\operatorname*{Lt}_{x\to 0}\frac{ae^{x}-b\cos x+ce^{-x}}{x\sin x}=2$$

Consider $\dfrac{ae^{x}-b\cos x+ce^{-x}}{x^2\left(\frac{\sin x}{x}\right)}$

(ii) Test the convergence of integral $\int_0^\infty \frac{\sin^2 x}{x^2}dx.$ **(16)**

① $\boxed{a-b+c=0}$ — ① Then Numerator $=0$.

$$\lim_{x\to 0}\frac{ae^{x}+b\sin x-ce^{-x}}{2x} \Rightarrow \boxed{a-c=0} \text{ — ②}$$

$$=\lim_{x\to 0}\frac{ae^{x}+b\cos x+ce^{-x}}{2}$$

$$=\frac{a+b+c}{2}=2$$

$$\Rightarrow \boxed{a+b+c=4} \text{ — ③}$$

solving ①, ② & ③

$a=1$, $b=2$, $c=1$

(ii) $$\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \int_0^1 \frac{\sin^2 x}{x^2}\,dx + \int_1^\infty \frac{\sin^2 x}{x^2}\,dx.$$

first integral is proper.

$\Rightarrow$ convergence of $\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$ at ∞.

$$\frac{\sin^2 x}{x^2} \le \frac{1}{x^2}$$

and $\int_1^\infty \frac{dx}{x^2}$ is convergent as $n=2>1$

$\therefore \int_1^\infty \frac{\sin^2 x}{x^2}\,dx$ is convergent

$\therefore \int_0^\infty \frac{\sin^2 x}{x^2}\,dx$ is convergent.

2. (c) The section of a cone with vertex at P and guiding curve $(x^2/a^2)+(y^2/b^2)=1, z=0$ by the plane $x=0$ is a rectangular hyperbola. Show that the locus of P is $(x^2/a^2)+\{(y^2+z^2)/b^2\}=1$. **(15)**

Let $P(\alpha, \beta, \gamma)$ be vertex of the cone.

eqn of any generator is

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = r \text{ (say)}.$$

This cuts $z=0$ plane at point

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = -\frac{\gamma}{n} \Rightarrow \begin{array}{l} x = \alpha - \frac{l}{n}\gamma \\ y = \beta - \frac{m}{n}\gamma . \end{array}$$

This lies on $x^2/a^2 + y^2/b^2 = 1$ iff

$$\frac{1}{a^2}\left(\alpha - \frac{l}{n}\gamma\right)^2 + \frac{1}{b^2}\left(\beta - \frac{m}{n}\gamma\right)^2 = 1.$$

eliminating l, m, n.

$$\frac{1}{a^2}\left(\alpha - \left(\frac{x-\alpha}{z-\gamma}\right)\gamma\right)^2 + \frac{1}{b^2}\left(\beta - \left(\frac{y-\beta}{z-\gamma}\right)\gamma\right)^2 = 1$$

section of this cone by plane $x=0$ is

$$\frac{1}{a^2}\left(\alpha + \frac{\alpha\gamma}{z-\gamma}\right)^2 + \frac{1}{b^2}\left(\beta - \left[\frac{y-\beta}{z-\gamma}\right]\gamma\right)^2 = 1.$$

$$\frac{\alpha^2}{a^2}\left(\frac{z}{z-\gamma}\right)^2 + \frac{1}{b^2}\left(\frac{\beta z - \gamma y}{z-\gamma}\right)^2 = 1.$$

$$\Rightarrow \alpha^2 b^2 z^2 + a^2(\beta z - \gamma y)^2 = (z-\gamma)^2 a^2 b^2$$

This is a rectangular hyperbola in $x=0$ plane if

coeff. y^2 + coeff. $z^2 = 0$

$\alpha^2 b^2 + a^2\beta^2 + a^2\gamma^2 = a^2 b^2$

$\therefore$ locus of $P(\alpha,\beta,\gamma)$ is

$b^2x^2 + a^2y^2 + a^2z^2 = a^2b^2$

$\Rightarrow \boxed{\dfrac{x^2}{a^2} + \dfrac{y^2+z^2}{b^2} = 1}$

3. (a) (i) Find a 2 x 2 real matrix A which is both orthogonal and skew-symmetric. Can there exist a 3 x 3 real matrix which is both orthogonal and skew-symmetric? Justify your answer.

(ii) Show that the matrix

$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ Satisfies Cayley-Hamilton theorem. Hence or otherwise obtain the value of A^{-1} and A^{-2}. **(20)**

SECTION-B

5. (a) Solve $4(x-2)^2\,(dy/dx) = (x+y-1)^2$. **(10)**

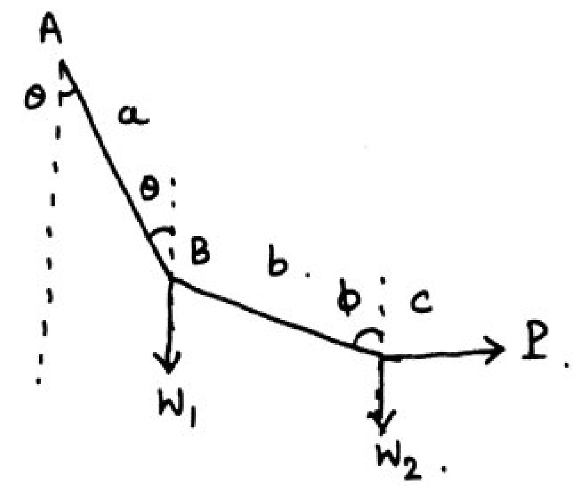
5. (b) Find the orthogonal trajectories of the system of circles touching a given straight line at a given point. (10)

5. (c) Weights W_1, W_2 are fastened to a light inextensible string ABC at the points B, C the end A being fixed. Prove that, if a horizontal force P is applied at C and equilibrium AB, BC are inclined at angles θ, ϕ to the vertical, then $P = (W_1 + W_2)\tan\theta = W_2 \tan\phi$. **(10)**

Let $AB = a$; $BC = b$.

given small displacement such that $\theta \to \theta + \delta\theta$

$\phi \to \phi + \delta\phi$.



But principle of virtual work

$$W_1(\delta a\cos\theta) + W_2\{\delta(a\cos\theta + b\cos\phi) + P\,\delta(a\sin\theta + b\sin\phi) = 0$$

$\Rightarrow -W_1 a \sin\theta\, \delta\theta - W_2 (a \sin\theta\, \delta\theta + b \sin\phi\, \delta\phi)$

$+ P (a \cos\theta\, \delta\theta + b \cos\phi\, \delta\phi) = 0$

as θ & ϕ are independent

i) keep ϕ fixed $\Rightarrow \delta\phi = 0$

$\Rightarrow -W_1 a \sin\theta\, \delta\theta - W_2 a \sin\theta\, \delta\theta + P a \cos\theta\, \delta\theta = 0$

$\Rightarrow W_1 \sin\theta + W_2 \sin\theta = P \cos\theta$

$\Rightarrow \boxed{P = (W_1 + W_2) \tan\theta}$

ii) keep θ fixed $\Rightarrow \delta\theta = 0$

$\Rightarrow -W_2 (b \sin\phi\, \delta\phi) + P b \cos\phi\, \delta\phi = 0$

$\Rightarrow P \cos\phi = W_2 \sin\phi$

$\Rightarrow \boxed{P = W_2 \tan\phi}$

$\therefore \boxed{P = (W_1 + W_2) \tan\theta = W_2 \tan\phi}$

5\. (d) A shell lying in a straight smooth horizontal tube suddenly breaks into two portions of masses m_1 and m_2. If s is the distance apart, in the tube, of the masses after a time t, show that the work done by the explosion is $\frac{1}{2}\frac{m_1 m_2}{m_1+m_2}\frac{s^2}{t^2}$ (08)

5. (e) (i) If A and B are given, prove that the line integral $\int_A^B (z^2\,dx + 2y\,dy + 2xz\,dz$ is independent of the path of integration.

(ii) Prove that $\int_V (g.\mathit{curl}\ \mathit{curl}\ f - f.\mathit{curl}\ \mathit{curl}\ g)dV = \int_S \{(f \times \mathit{curl}\ g) - (g \times \mathit{curl}\ f\}.da$ [16.

(i)

$\int_A^B \vec{F}\cdot d\vec{r}$ where $\vec{F} = z^2\hat{i} + 2y\hat{j} + 2xz\hat{k}$

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ z^2 & 2y & 2xz \end{vmatrix}$$

$= \hat{i}(0) - \hat{j}(2z - 2z) + \hat{k}(0)$

$= 0$

$\therefore$ curl $\vec{F} = 0$

$\Rightarrow \vec{F}$ is conservative force

$\Rightarrow \int_A^B \vec{F}\cdot d\vec{r}$ does not depend of path of integration.

(ii) $\text{div}(\vec{f}\times\vec{g}) = \vec{g}\cdot \text{curl}\,\vec{f} - \vec{f}\cdot\text{curl}\,\vec{g}$

$\text{div}(\vec{f}\times \text{curl}\,\vec{g}) = \text{curl}\,\vec{g}\cdot\text{curl}\,\vec{f} - \vec{f}\cdot\text{curl}(\text{curl}\,\vec{g})$

$\text{div}(\vec{g}\times\text{curl}\,\vec{f}) = \text{curl}\,\vec{f}\cdot\text{curl}\,\vec{g}$
$\quad - \vec{g}\cdot\text{curl}(\text{curl}\,\vec{f})$

$\therefore \operatorname{div}(\vec{f} \times \operatorname{curl} \vec{g}) - \operatorname{div}(\vec{g} \times \operatorname{curl} \vec{f})$

$= \vec{g} \cdot \operatorname{curl}(\operatorname{curl} \vec{f}) - \vec{f} \cdot \operatorname{curl}(\operatorname{curl} \vec{g})$ — ①

$\therefore \int_S \vec{A} \cdot d\vec{a} = \int_V \operatorname{div} \vec{A} \, dV$

$\Rightarrow \int_S \{(\vec{f} \times \operatorname{curl} \vec{g}) - \vec{g} \times \operatorname{curl} \vec{f}\} \cdot d\vec{a}$

$= \int_V \{\operatorname{div}(\vec{f} \times \operatorname{curl} \vec{g}) - \operatorname{div}(\vec{g} \times \operatorname{curl} \vec{f})\} \, dV$

$= \int_V \{\vec{g} \cdot \operatorname{curl}(\operatorname{curl} \vec{f}) - \vec{f} \cdot \operatorname{curl}(\operatorname{curl} \vec{g})\} \, dV$

using ① .

6. (a) Solve $x^3(d^3y/dx^3) + 2x^2(d^2y/dx^2) + 2y = 10(x + 1/x)$. **(10)**

P.T.O.

7. (a) Show that the length of an endless chain which will hang over a circular pulley of radius a so as to be in contact with two-thirds of the circumference of the pulley is

$$a\left\{\frac{3}{\log\,(2+\sqrt{3}}+\frac{4\pi}{3}\right\}$$ **(18)**

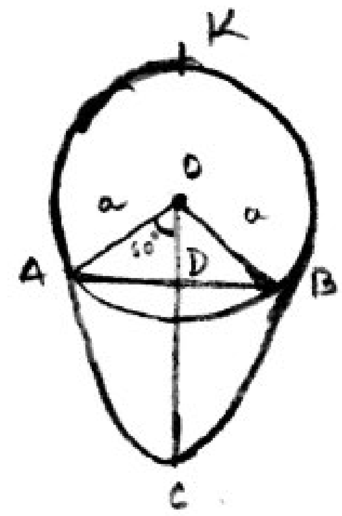
Tangent at B makes an angle $\pi/3$ with horizontal $\Rightarrow$ $\psi_B = \pi/3$.

$BC = s = c\tan\psi_B = c\cdot\sqrt{3}$.

$BD = a\sin\pi/3 = \frac{\sqrt{3}}{2}a$.

But $x = c\log(\sec\psi + \tan\psi)$

$BD = x_B = c\log(\sec\psi_B + \tan\psi_B)$



$$BD = c \log\left(2 + \sqrt{3}\right) = \frac{\sqrt{3}}{2} a$$

$$\Rightarrow c = \frac{\sqrt{3}\, a}{2 \log(2+\sqrt{3})}$$

$\therefore$ arc ACB $= 2$ arc BC $= 2c\sqrt{3}$

$$= \frac{3a}{\log(2+\sqrt{3})}.$$

arc AKB $= \frac{2}{3}(2\pi a) = \frac{4\pi a}{3}$

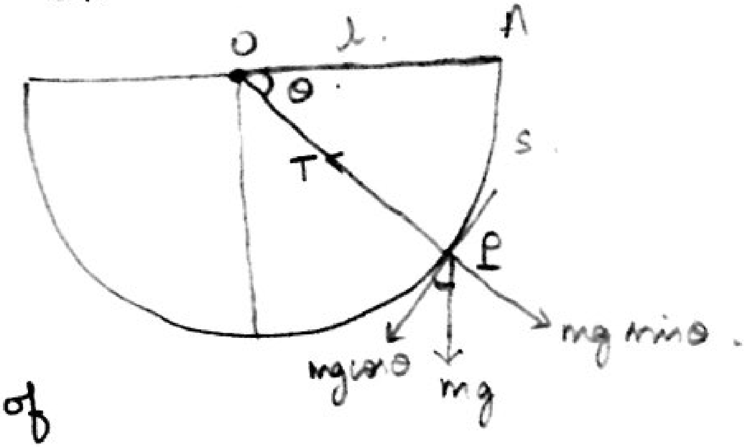
$\therefore$ length of the chain $=$ arc ACB $+$ arc AKB

$$= \frac{4\pi}{3} a + \frac{3a}{\log(2+\sqrt{3})}$$

$$= a\left\{ \frac{4\pi}{3} + \frac{3}{\log(2+\sqrt{3})} \right\}.$$

P.T.O.

7. (b) A particle attached to a fixed peg O by string of length l, is lifted up with the string hotizontal and then let go. Prove that when the string makes an angle θ with the hotizontal, the resultant acceleration is $g\sqrt{(1+3\sin^2\theta)}$. **(18)**



eqn of motion of the particle at P.

$$m\frac{d^2s}{dt^2} = mg\cos\theta$$

$$m\frac{v^2}{l} = T - mg\sin\theta.$$

also $s = l\theta$.

$$\Rightarrow \frac{d^2s}{dt^2} = l\frac{d^2\theta}{dt^2}$$

$$\Rightarrow l\frac{d^2\theta}{dt^2} = g\cos\theta.$$; multiply by $2l\frac{d\theta}{dt}$ and integrating

$$v^2 = \left(l\frac{d\theta}{dt}\right)^2 = 2gl\sin\theta.$$

as $\theta = 0$, $v = 0$ $\Rightarrow$ constant of Integration $= 0$

resultant acceleration

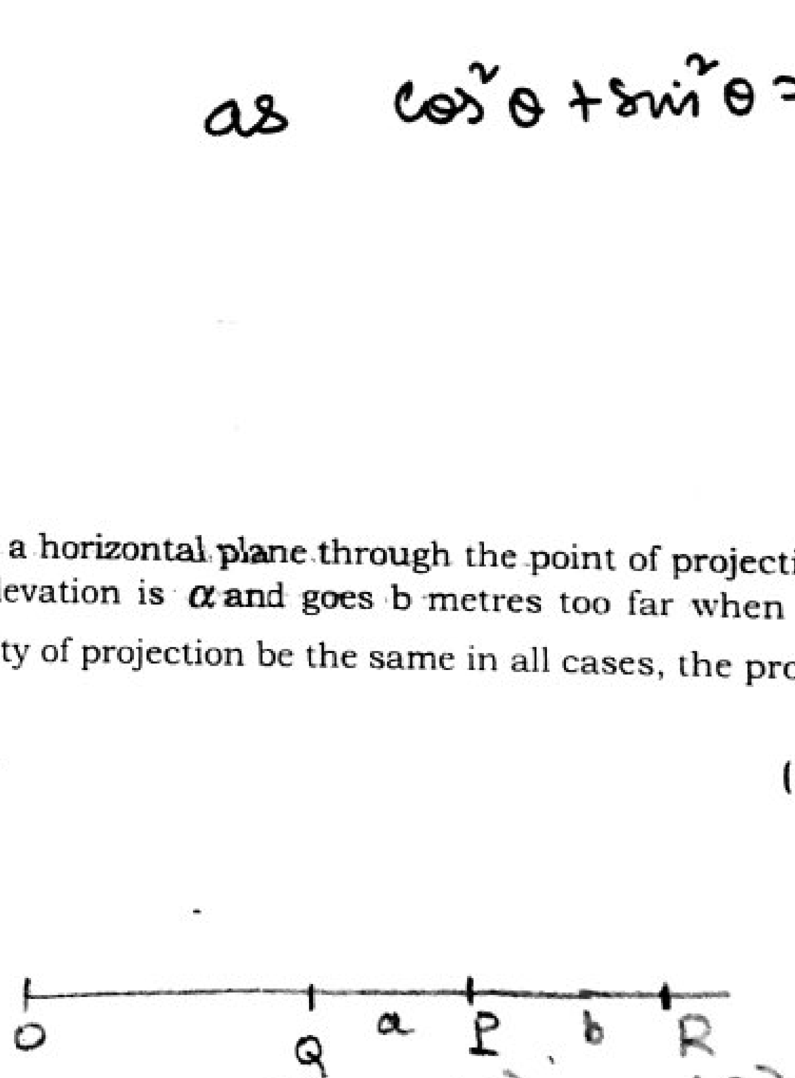
$$= \sqrt{\left(\frac{d^2s}{dt^2}\right)^2 + \left(\frac{v^2}{l}\right)^2}$$

$$= \sqrt{(g\cos\theta)^2 + (2g\sin\theta)^2}$$

$$= g\sqrt{\cos^2\theta + 4\sin^2\theta}$$

$$= g\sqrt{1 + 3\sin^2\theta}. \quad \text{as } \cos^2\theta + \sin^2\theta = 1$$

7. (c) A projectile aimed at mark which is in a horizontal plane through the point of projection, falls a metres short of it when the elevation is α and goes b metres too far when the elevation is β. Show that, if the velocity of projection be the same in all cases, the proper elevation is $\frac{1}{2}\sin^{-1}\frac{a\sin 2\beta + b\sin 2\alpha}{a+b}$. **(14)**



Range $R = \frac{u^2 \sin 2\alpha}{g}$.

$OQ = \frac{u^2\sin 2\alpha}{g}$; $OP = \frac{u^2 \sin 2\theta}{g}$

$OR = \frac{u^2 \sin 2\beta}{g}$.

$$\left\{\begin{array}{l} OP = OQ + a \ ; \quad OR = OP + b . \\ \Rightarrow OP = OR - b . \end{array}\right.$$

$$\Rightarrow bOP + aOP = b(OQ + a) + a(OR - b)$$

$$\Rightarrow (a+b)OP = bOQ + aOR$$

$$\therefore \ (a+b)\frac{u^2}{g}\sin 2\theta = b\frac{u^2}{g}\sin 2\alpha + a\frac{u^2}{g}\sin 2\beta$$

$$\Rightarrow (a+b)\sin 2\theta = b\sin 2\alpha + a\sin 2\beta .$$

$$\Rightarrow \sin 2\theta = \frac{b\sin 2\alpha + a\sin 2\beta}{a+b}$$

$$\boxed{\theta = \frac{1}{2}\sin^{-1}\left\{\frac{b\sin 2\alpha + a\sin 2\beta}{a+b}\right\}}$$

8. (a) If $F = \mathbf{i}x^2 + \mathbf{j}yz + \mathbf{k}y^2$, compute $\int_A^B F.dR$, where A=(0, 0, 0), B= (0, 3, 4), along the straight line connecting these points. **(08)**

eqn of AB $\Rightarrow \dfrac{x}{0} = \dfrac{y}{3} = \dfrac{z}{4} = t$ (say)

$\Rightarrow x = 0;\ y = 3t;\ z = 4t.$

$\vec{F} = \hat{j}(12t^2) + \hat{k}(9t^2)$

$\int_A^B \vec{F}\cdot d\vec{r} = \int_{t=0}^{1} (12t^2)\,3\,dt + (9t^2)\,4\,dt.$

$= \int_{t=0}^{1} 72t^2\,dt = \dfrac{72}{3} = \underline{\underline{24}}.$

(b) (i) Find the values of the constants a, b, c so that the directional derivative of $\phi = ax^2 + by^2 + cz^2$ at (1, 1, 2) has a maximum magnitude 4 in the direction parallel to y-axis. **(14)**

(ii) Prove that $\nabla^2\left[\nabla\cdot\left(\frac{\vec{r}}{r^2}\right)\right] = 2r^{-4}$

$\nabla\phi = \hat{i}(2ax) + \hat{j}(2by) + \hat{k}(2cz)$.

at (1, 1, 2)

$\nabla\phi = \hat{i}(2a) + \hat{j}(2b) + \hat{k}(4c)$.

$|\nabla\phi|^2 = 4a^2 + 4b^2 + 16c^2 = 16$

$\Rightarrow \underline{a^2 + b^2 + 4c^2 = 4}$.

Also $\Rightarrow a + b + 2c = 0$.

and $\dfrac{\nabla\phi}{|\nabla\phi|} = \hat{j}$

$\Rightarrow \dfrac{2b}{\sqrt{4a^2+4b^2+16c^2}} = 1$

$\Rightarrow b^2 = a^2 + b^2 + 4c^2$

$\Rightarrow \underline{a^2 + 4c^2 = 0}$ $\Rightarrow$ $b^2 = 4$

$\underline{b = \pm 2}$

$$\nabla\cdot(\phi\vec{A}) = \nabla\phi\cdot\vec{A} + \phi\,\nabla\cdot\vec{A}$$

$$\phi = \frac{1}{r^2}; \quad \vec{A} = \vec{r}.$$

$$\nabla\phi = \sum i\,\frac{\partial\phi}{\partial x} = \sum \hat{i}\,\frac{\partial}{\partial x}\left(\frac{1}{r^2}\right) = \sum \hat{i}\left(\frac{-2}{r^3}\right)\frac{x}{r}$$

$$= \sum -\frac{2}{r^4}\,x\hat{i} = -\frac{2}{r^4}\vec{r}.$$

$$\nabla\cdot\vec{A} = \nabla\cdot\vec{r} = 3.$$

$$\therefore \nabla\cdot\left(\frac{1}{r^2}\vec{r}\right) = -\frac{2}{r^4}\vec{r}\cdot\vec{r} + \frac{1}{r^2}\cdot 3 = \frac{1}{r^2}.$$

$$\frac{\partial}{\partial x}\left(\frac{1}{r^2}\right) = \frac{-2}{r^3}\frac{\partial r}{\partial x} = -\frac{2}{r^3}\frac{x}{r} = -\frac{2x}{r^4}.$$

$$\frac{\partial^2}{\partial x^2}\left(\frac{1}{r^2}\right) = -2\,\frac{\partial}{\partial x}\left(\frac{x}{r^4}\right) = -2\left[\frac{1}{r^4} + x\left(\frac{-4}{r^5}\right)\frac{x}{r}\right]$$

$$= -2\left[\frac{1}{r^4} - 4\frac{x^2}{r^6}\right]$$

$$\therefore \nabla^2\left[\nabla\cdot\left(\frac{\vec{r}}{r^2}\right)\right] = \sum\frac{\partial^2}{\partial x^2}\left(\frac{1}{r^2}\right)$$

$$= -2\left[\frac{3}{r^4} - \frac{4}{r^6}\left(x^2+y^2+z^2\right)\right] = -2\left[\frac{3}{r^4} - \frac{4}{r^4}\right]$$

$$= \frac{2}{r^4} = \underline{\underline{2r^{-4}}}.$$

(c) Find κ *and* τ for the space curve $x=t,\ y=t^2,\ t=t^3$. **(10)**

$$r = t\hat{i} + t^2\hat{j} + t^3\hat{k}$$

$$\dot{r} = \hat{i} + 2t\hat{j} + 3t^2\hat{k}$$

$$\ddot{r} = 2\hat{j} + 6t\hat{k}$$

$$\dddot{r} = 6\hat{k}.$$

$$\dot{r}\times\ddot{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} = \hat{i}(6t^2) - \hat{j}(6t) + \hat{k}(2)$$

$$|\dot{r}\times\ddot{r}| = \sqrt{36t^4 + 36t^2 + 4}$$

$$[\dot{r}\ \ddot{r}\ \dddot{r}] = (\dot{r}\times\ddot{r})\cdot\dddot{r} = 12$$

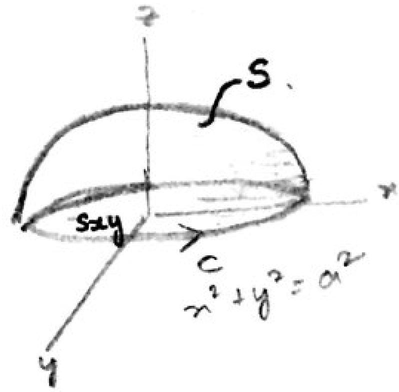
$$K = \frac{|\dot{r}\times\ddot{r}|}{|\dot{r}|^3} = \frac{2\sqrt{9t^4+9t^2+1}}{(1+4t^2+9t^4)^{3/2}}$$

$$\tau = \frac{[\dot{r}\ \ddot{r}\ \dddot{r}]}{|\dot{r}\times\ddot{r}|^2} = \frac{12}{4(9t^4+9t^2+1)}$$

$$\tau = \frac{3}{9t^4+9t^2+1}$$

(d) Verify Stoke's theorem for the vector **F**=z **i** + x **j** + y **k** taken over the half of the sphere $x^2+y^2+z^2=a^2$ lying above the xy-plane. **(18)**

$$\int_C \vec{F}\cdot dr = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,dS.$$



on C $\Rightarrow$ $z=0$; $dz=0$.

$$\vec{F} = x\hat{j} + y\hat{k}$$

$$\int_C \vec{F}\cdot d\vec{r} = \int (x\hat{j}+y\hat{k})\cdot(dx\,\hat{i}+dy\,\hat{j})$$

$x = a\cos\theta$

$y = a\sin\theta$

$$= \int x\,dy$$

$$= \int_{\theta=0}^{2\pi} a\cos\theta\cdot a\cos\theta\,d\theta = a^2\int_0^{2\pi}\cos^2\theta\,d\theta$$

$$= a^2\cdot 4\int_0^{\pi/2}\cos^2\theta\,d\theta = a^2\cdot 4\cdot\frac{1}{2}\,\frac{\pi}{2} = a^2\pi \quad \text{— (1)}$$

$$\text{curl}\,\vec{F} = \hat{i}+\hat{j}+\hat{k}$$

let Stot be total closed surface consisting of S & Sxy.

$$\therefore \iint_{S_{tot}} \text{curl}\,\vec{F}\cdot\hat{n}\,ds = \iiint_V \text{div}(\text{curl}\,\vec{F})\,dv = 0$$

as $\text{div}(\text{curl}\,\vec{F}) = 0$

P.T.O.

$$\Rightarrow \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds + \iint_{S_{xy}} \text{curl}\,\vec{F}\cdot\hat{n}\,ds = 0$$

$$\Rightarrow \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds = -\iint_{S_{xy}} \text{curl}\,\vec{F}\cdot\hat{n}\,ds.$$

$S_{xy} \Rightarrow x^2+y^2 \le a^2$; $\hat{n} = -\hat{k}$; $ds = dx\,dy$

as $z=0$.

$\text{curl}\,\vec{F}\cdot\hat{n} = -1$.

$$\therefore \iint_{S_{xy}} \text{curl}\,\vec{F}\cdot\hat{n}\,ds = \iint_{S_{xy}} (-1)\,dx\,dy$$

$$= -\pi a^2$$

$$\therefore \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds = \pi a^2 \quad \text{--- (2)}$$

from (1) & (2)

$$\boxed{\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,dS}$$

Hence stokes theorem is verified.

P.T.O.