A CONSOLIDATED QUESTION PAPER-CUM-ANSWER BOOKLET

IMS™

(INSTITUTE OF MATHEMATICAL SCIENCES)

TEST SERIES (MAIN)-2014

Test Code: FULL TEST P-I(M) IAS / T- 12

MATHEMATICS

by K. VENKANNA

The person with 14 years of Teaching Experience

FULL TEST P-II

Time: Three Hours | *Maximum Marks: 250*

INSTRUCTIONS

1. This question paper-cum-answer booklet has 52 pages and has 34 PART/SUBPART questions. Please ensure that the copy of the question paper-cum-answer booklet you have received contains all the questions.
2. Write your Name, Roll Number, Name of the Test Centre and Medium in the appropriate space provided on the right side.
3. A consolidated Question Paper-cum-Answer Booklet, having space below each part/sub part of a question shall be provided to them for writing the answers. Candidates shall be required to attempt answer to the part/sub-part of a question strictly within the pre-defined space. Any attempt outside the pre-defined space shall not be evaluated."
4. Answer must be written in the medium specified in the admission Certificate issued to you, which must be stated clearly on the right side. No marks will be given for the answers written in a medium other than that specified in the Admission Certificate.
5. Candidates should attempt Question Nos. 1 and 5, which are compulsory, and any **THREE** of the remaining questions selecting at least **ONE** question from each Section.
6. The number of marks carried by each question is indicated at the end of the question. Assume suitable data if considered necessary and indicate the same clearly.
7. Symbols/notations carry their usual meanings, unless otherwise indicated.
8. All questions carry equal marks.
9. All answers must be written in blue/black ink only. Sketch pen, pencil or ink of any other colour should not be used.
10. All rough work should be done in the space provided and scored out finally.
11. The candidate should respect the instructions given by the invigilator.
12. The question paper-cum-answer booklet must be returned in its entirety to the invigilator before leaving the examination hall. Do not remove any page from this booklet.

READ INSTRUCTIONS ON THE LEFT SIDE OF THIS PAGE CAREFULLY

Name: Nitish.K

Roll No.: 149709

Test Centre: Bangalore

Medium: English

Do not write your Roll Number or Name anywhere else in this Question Paper-cum-Answer Booklet.

I have read all the instructions and shall abide by them.

Nitish.K

Signature of the Candidate

I have verified the information filled by the candidate above

Signature of the invigilator

IMPORTANT NOTE:

Whenever a question is being attempted, all its parts/ sub-parts must be attempted contiguously. This means that before moving on to the next question to be attempted, candidates must finish attempting all parts/ sub-parts of the previous question attempted. This is to be strictly followed. Pages left blank in the answer-book are to be clearly struck out in ink. Any answers that follow pages left blank may not be given credit.

P.T.O.

INDEX TABLE

QUESTION	No.	PAGE NO.	MAX. MARKS	MARKS OBTAINED
1	(a)			
	(b)			
	(c)			
	(d)			
	(e)			
2	(a)			
	(b)			
	(c)			
	(d)			
3	(a)			
	(b)			
	(c)			
	(d)			
4	(a)			
	(b)			
	(c)			
	(d)			
5	(a)			
	(b)			
	(c)			
	(d)			
	(e)			
6	(a)			
	(b)			
	(c)			
	(d)			
7	(a)			
	(b)			
	(c)			
	(d)			
8	(a)			
	(b)			
	(c)			
	(d)			
			Total Marks	

SECTION-A

1. (a) In Z_{24}, find a generator for $\langle 21\rangle \cap \langle 10\rangle$. Suppose that $|a| = 24$. Find a generator for $\langle a^{21}\rangle \cap \langle a^{10}\rangle$. In general, what is a generator for the subgroup $\langle a^{m}\rangle \cap \langle a^{n}\rangle$? **(10)**

$$\langle 21\rangle \cap \langle 10\rangle = \langle \gcd(21, 10)\rangle = \langle 1\rangle$$

$$\therefore \langle 21\rangle \cap \langle 10\rangle = \mathbb{Z}_{24}.$$

$$\Rightarrow \boxed{\langle a^{m}\rangle \cap \langle a^{n}\rangle = \langle a^{\gcd(m,n)}\rangle}$$

$$\therefore \langle a^{21}\rangle \cap \langle a^{10}\rangle = \langle a^{\gcd(21,10)}\rangle = \langle a^{1}\rangle$$

$$\therefore \underline{\langle a^{21}\rangle \cap \langle a^{10}\rangle = \langle a\rangle}.$$

1. (b) Show that the mapping $f:\mathbb{C}\to M_2(\mathbb{R})$ defined by $f(a+ib)=\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ is a homomorphism of rings. Find $ker f$. Is f a monomorphism? Is f an isomorphism of $\mathbb{C}$ onto $M_2(\mathbb{R})$? **(10)**

$$f[(a+ib)(c+di)] = f((ac-bd)+i(bc+ad))$$

$$= \begin{bmatrix} ac-bd & bc+ad \\ -bc-ad & ac-bd \end{bmatrix}$$

$$f(a+ib)f(c+di) = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}\begin{bmatrix} c & d \\ -d & c \end{bmatrix}$$

$$= \begin{bmatrix} ac-bd & ad+bc \\ -bc-ad & ac-bd \end{bmatrix}$$

$$\therefore f[(a+bi)(c+di)] = f(a+bi)\, f(c+di).$$

$$f[(a+bi)+(c+di)] = f[(a+c)+(b+d)i]$$

$$= \begin{bmatrix} a+c & b+d \\ -b-d & a+c \end{bmatrix}$$

$$f(a+bi)+f(c+di) = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} + \begin{bmatrix} c & d \\ -d & c \end{bmatrix}$$

$$= \begin{bmatrix} a+c & b+d \\ -b-d & a+c \end{bmatrix}$$

$$\Rightarrow f[(a+bi)+(c+di)] = f(a+bi)+f(c+di).$$

if $a+bi \in \ker f \Leftrightarrow f(a+bi) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$\Leftrightarrow \begin{bmatrix} a & b \\ -b & a \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Leftrightarrow a=0, b=0$$

$\therefore$ ~~ker f = {0}~~ $\boxed{\ker f = \{0\}}$

$\therefore f$ is 1-1 $\Rightarrow f$ is monomorphism.

No f is not an isomorphism from C onto $M_2(R)$ as f is not ONTO.

as $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{R})$ doesnot have pre-image.

1. (c) For a real a, show that if $f:\mathbb{R}\rightarrow\mathbb{R}$ is a continuous solution of the equation

$f(x+y) = f(x)+f(y)+axy$, then $f(x) = \frac{a}{2}x^2 + bx, where b = f(1) - \frac{a}{2}$. **(10)**

~~$f(1+1) = f(1) + f(1) + a$~~

~~$= 2f(1) + \frac{a}{2} + \frac{a}{2}$~~

~~$= \frac{a}{2}$~~

1. (d) Show that the function $f(z)=e^{-z^{-4}}(z\neq 0)$ and $f(0)=0$ is not analytic at z=0, although Cauchy-Riemann equations are satisfied at the point. How would you explain this. **(10)**

$f(z) = u + iv$, where

$$u = e^{-\frac{1}{r^8}(x^4+y^4-6x^2y^2)} \cos\left(\frac{4xy}{r^6}\right)$$

$$v = e^{-\frac{1}{r^8}(x^4+y^4-6x^2y^2)} \sin\left(\frac{4xy}{r^6}\right)$$

$$u_x(0,0) = \lim_{h\to 0} \frac{u(h,0)-u(0,0)}{h} = \lim_{h\to 0} \frac{e^{-1/h^4}}{h}$$

$$= \lim_{h\to 0} \frac{1}{he^{1/h^4}} = \lim_{h\to 0} \frac{1}{h\left[1+\frac{1}{h^4}+\frac{1}{2h^8}+\cdots\right]}$$

$$= 0 .$$

$$u_y(0,0) = \lim_{k\to 0}\frac{u(0,k)-u(0,0)}{k} = \lim_{k\to 0}\frac{e^{-1/k^4}}{k} = 0$$

Also $v_x(0,0) = 0$ & $v_y(0,0) = 0$

$\therefore$ $\underline{u_x = v_y \text{ \& } v_x = -u_y}$ at $(0,0)$.

$$f'(0) = \lim_{z\to 0}\frac{f(z)-f(0)}{z} = \lim_{z\to 0}\frac{e^{-z^{-4}}}{z} ; \quad \boxed{\text{as } z\to 0 \quad e^{-z^{-4}} \approx 1-z^{-4}}$$

$$= \lim_{z\to 0}\frac{1-z^{-4}}{z} = \lim_{z\to 0}\frac{z^4-1}{z^5} \to \infty$$

$\therefore f'(0)$ doesnot exist $\Rightarrow f(z)$ is not analytic at $z=0$.

1. (e) Show that although $(2,3,2)^T$ is a feasible solution to the system of equations

$$x_1 + x_2 + 2x_3 = 9$$
$$3x_1 + 2x_2 + 5x_3 = 22$$
$$x_1, \; x_2, \; x_3 \geq 0,$$

it is not a basic solution. How many basic solutions this system may have? Find all the basic feasible solutions of the system. **(10)**

$n = 3$; $m = 2$

for basic soln $\Rightarrow$ $n - m = 1$ non-basic variable <u>must be zero</u>.

but $(2,3,2)^T$ does not have any non basic variable ~~with~~ which is zero

$\therefore$ $(2,3,2)^T$ cannot be basic.

No of basic solutions $= {}^nC_m = {}^3C_2 = 3$.

non basic variable	basic variable	basic solution.	Is feasible
x_1	x_2, x_3	$x_2 = 1$, $x_3 = 4$	Yes
x_2	x_1, x_3	$x_1 = -1$, $x_3 = 5$	No
x_3	x_1, x_2	$x_1 = 4$, $x_2 = 5$	Yes

$\therefore$ basic feasible solutions are

$(0, 1, 4)$ & $(4, 5, 0)$

2. (a) (i) In Z_{24}, list all generators for the subgroup of order 8. Let $G = \langle a \rangle$ and let $|a| = 24$. List all generators for the subgroup of order 8.

(ii) In Z, find all generators of the subgroup $\langle 3 \rangle$. If a has order, find all generators of the subgroup $\langle a^3 \rangle$. **(18)**

(i) There is only one subgroup of order 8

$\Rightarrow \langle 3 \rangle = \{0, 3, 6, 9, 12, 15, 18, 21\}$.

d will be a generator iff $\gcd(24, d) = 3$.

$\Rightarrow d = 9, 15, 21$.

$\therefore$ generators are 3, 9, 15, 21.

Generators of subgroup of order 8 in $G = \langle a \rangle$; $|a| = 24$.

gcd(24.

$H = \langle a^3 \rangle$ is a subgroup of order 8.

The generators are a^3, a^9, a^{15}, a^{21}.

(ii) $\langle 3 \rangle = 3\mathbb{Z}$

$= \{\ldots, -6, -3, 0, 3, 6, 9, 12, \ldots$

clearly only generators are $\underline{3 \; \& \; -3}$

2. (b) (i) Prove that every infinite bounded set of real numbers has a limit point.

(ii) Let [x] denote the integer part of the real number x, i.e. if $n \leq x < n+1$ where n is an integer, then $[x] = n$. Is the function $f(x) = [x]^2 + 3$ Riemann integrable in $[-1, 2]$?

If not, explain why. If it is integrable, compute $\int_{-1}^{2}\left([x]^2+3\right)dx$. **(14)**

(ii) $$f(x) = \begin{cases} 4 ; & -1 \leq x < 0 . \\ 3 ; & 0 \leq x \leq 1 \\ 4 ; & 1 \leq x < 2 \\ 7 ; & x = 2 \end{cases}$$

clearly the only points of discontinuities are. $x = 0, 1, 2$; which are finite in number.

$\Rightarrow$ points of discontinuity are finite

$\Rightarrow f$ is Reimann integrable.

$$\int_{-1}^{2} f(x)\,dx = \int_{-1}^{0} 4\,dx + \int_{0}^{1} 3\,dx + \int_{1}^{2} 4\,dx$$

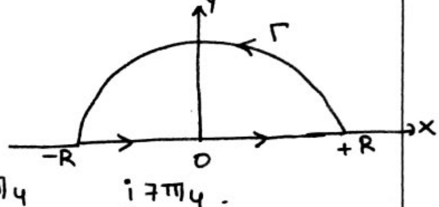
$$= 4(0-(-1)) + 3(1-0) + 4(2-1)$$

$$= 4 + 3 + 4 = \underline{\underline{11}}$$

P.T.O.

2. (c) Use the method contour integration to prove that $\int_0^{\infty} \frac{x^6 dx}{(a^4+x^4)^2} = \frac{3\pi\sqrt{2}}{16a}, a>0$ **(18)**

Consider $\int_C f(z)\,dz$ where $f(z) = \frac{z^6}{(a^4+z^4)^2}$.



poles of $f(z)$ is

$z^4 + a^4 = 0$.

$z = ae^{i\pi/4}, ae^{i3\pi/4}, ae^{i5\pi/4}, ae^{i7\pi/4}$.

only the first Two lies within C.

let $\alpha = ae^{i\pi/4}$; $\beta = ae^{i3\pi/4}$.

Residue at $z=\alpha$. : Put $z-\alpha = t$ or $\underline{z = \alpha + t}$

$$f(z) = \frac{(\alpha+t)^6}{[a^4+(\alpha+t)^4]^2} = \frac{(\alpha+t)^6}{(a^4+\alpha^4+4\alpha^3 t+6\alpha^2 t^2+\cdots)^2}$$

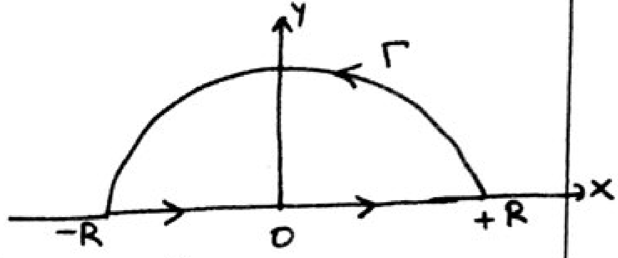
$$= \frac{1}{16\alpha^6 t^2}[\alpha+t]^6\left[1+\frac{3}{2\alpha}t\right]^{-2}.$$

$$= \frac{1}{16\alpha^6 t^2}\left[\alpha^6+6\alpha^5 t+\cdots\right]\left[1-\frac{3}{\alpha}t+\cdots\right]$$

$$= \frac{1}{16\alpha^6 t^2}\left[6\alpha^5 - 3\alpha^5\right]t + \cdots$$

2. (c) Use the method contour integration to prove that $\int_0^\infty \frac{x^6 dx}{(a^4+x^4)^2} = \frac{3\pi\sqrt{2}}{16a}, a>0$ **(18)**

consider $\int_C f(z)\,dz$ where $f(z) = \frac{z^6}{(a^4+z^4)^2}$.



poles of $f(z)$ is

$z^4 + a^4 = 0$.

$z = ae^{i\pi/4}, \ ae^{i3\pi/4}, \ ae^{i5\pi/4}, \ ae^{i7\pi/4}$.

only the first two lies within C.

let $\alpha = ae^{i\pi/4}$; $\beta = ae^{i3\pi/4}$.

Residue at $z=\alpha$. : Put $z-\alpha = t$ or $z = \alpha + t$

$$f(z) = \frac{(\alpha+t)^6}{[a^4+(\alpha+t)^4]^2} = \frac{(\alpha+t)^6}{(a^4+\alpha^4+4\alpha^3t+6\alpha^2t^2+\cdots)^2}$$

$$= \frac{1}{16\alpha^6t^2}[\alpha+t]^6\left[1+\frac{3}{2\alpha}t\right]^{-2}.$$

$$= \frac{1}{16\alpha^6t^2}[\alpha^6+6\alpha^5t+\cdots]\left[1-\frac{3}{\alpha}t+\cdots\right]$$

$$= \frac{1}{16\alpha^6t^2}\left[6\alpha^5-3\alpha^5\right]t+\cdots$$

$$= \frac{3\alpha^5}{16\alpha^6}\frac{1}{t} + \cdots = \frac{3}{16\alpha}\frac{1}{t} + \cdots$$

$$\Rightarrow \therefore \operatorname{Res}(z=\alpha) = \frac{3}{16\alpha} \;\&\; \operatorname{Res}(z=\beta) = \frac{3}{16\beta}$$

$$\Sigma R^+ = \frac{3}{16}\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) = \frac{3}{16a}\left(-\sqrt{2}\, i\right)$$

$$\therefore \int_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{\Gamma} f(z)\,dz = 2\pi i \Sigma R^+$$

$$\because \lim_{z\to\infty} z f(z) = 0 \Rightarrow \lim_{R\to\infty}\int_{\Gamma} f(z)\,dz = 0$$

$$\therefore \int_{-\infty}^{\infty} f(x)\,dx = 2\pi i\left(\frac{3}{16a}\right)\left(-\sqrt{2}\, i\right)$$

$$\Rightarrow \int_{0}^{\infty} f(x)\,dx = \pi\left(\frac{3}{16a}\right)\sqrt{2}$$

$$\boxed{\int_0^{\infty} \frac{x^6}{(a^4+x^4)^2}\,dx = \frac{3\pi\sqrt{2}}{16a}}$$

P.T.O.

SECTION-B

5. (a) Find the equation of the integral surface of the differential equation $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$ which passes through the line $x = 1, y = 0$. **(10)**

$$\frac{dx}{x^2-yz} = \frac{dy}{y^2-zx} = \frac{dz}{z^2-xy}.$$

$$\frac{d(x-y)}{(x-y)(x+y+z)} = \frac{d(y-z)}{(y-z)(x+y+z)} \Rightarrow \frac{d(x-y)}{x-y} = \frac{d(y-z)}{y-z}$$

$$\Rightarrow \log\left(\frac{x-y}{y-z}\right) = \log c_1 \Rightarrow \boxed{\frac{x-y}{y-z} = c_1} \quad -①$$

$$\frac{dx+dy+dz}{x^2+y^2+z^2-xy-yz-zx} = \frac{x\,dx+y\,dy+z\,dz}{x^3+y^3+z^3-3xyz}$$

$$d(x+y+z) = \frac{x\,dx+y\,dy+z\,dz}{x+y+z}.$$

$$\Rightarrow (x+y+z)\,d(x+y+z) = x\,dx+y\,dy+z\,dz.$$

$$(x+y+z)^2 = x^2+y^2+z^2+c_2'. \quad \text{where } c_2 = \frac{c_2'}{2}.$$

$$\Rightarrow \boxed{xy+yz+zx = c_2} \quad -②.$$

$$x=1;\ y=0 \Rightarrow \frac{1}{-z} = c_1 \ \& \ z = c_2.$$

$$\therefore \boxed{-1 = c_1 c_2}$$

$$\therefore \boxed{\left(\frac{x-y}{y-z}\right)(xy+yz+zx) = -1}$$

is the required integral surface.

5. (b) Solve $(D_x^3 - 7D_xD_y^2 - 6D_y^3)z = \sin(x+2y) + e^{3x-y}$. **(10)**

let $D_x = D$
$D_y = D'$

$$(D^3 - 7DD'^2 - 6D'^3)z = \sin(x+2y) + e^{3x-y}.$$

$$\therefore CF = \phi_1(y+3x) + \phi_2(y-x) + \phi_3(y-2x).$$

$$PI_1 = \frac{1}{(D-3D')(D+D')(D+2D')} e^{3x-y}$$

$$= \frac{1}{(3+3)(3-1)(3-2)} e^{3x-y} = \frac{e^{3x-y}}{12}$$

$$PI_2 = \frac{1}{D^3 - 7DD'^2 - 6D'^3} \sin(x+2y).$$

$D^2 \to -1$
$D'^2 \to -4$.
$DD' \to -2$

$$= \frac{1}{-D - 7D(-4) - 6D'(-4)} \sin(x+2y)$$

$$= \frac{1}{27D + 24D'} \sin(x+2y)$$

$$= \frac{(27D - 24D')\sin(x+2y)}{27^2D^2 - 24^2D'^2} = \frac{27\cos(x+2y) - 48\cos(x+2y)}{1575}.$$

$$= -\frac{1}{75}\cos(x+2y)$$

$$\therefore z = \phi_1(y+3x) + \phi_2(y-x) + \phi_3(y-2x) + \frac{1}{12}e^{3x-y} - \frac{1}{75}\cos(x+2y)$$

5. (c) Using Newton's iterative method, find the real root of $x \log_{10} x = 1.2$ correct to five decimal places. **(10)**

$f(x) = x \log_{10} x - 1.2$

let $\underline{x_0 = 2.}$

$f'(x) = \frac{1}{\ln 10}[1 + \ln x]$

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

$\therefore x_1 = 2.81316$

$x_2 = 2.74110$

$x_3 = 2.74065$

$x_4 = 2.74065$

$x_5 = 2.74065$

$\therefore \underline{x = 2.74065}$ upto 5 decimal places.

5. (d) A majority function is a digit circuit whose output is '1' iff the majority of the inputs are 1. The output is '0' otherwise. Obtain the truth table of a three-input majority function and show that the circuit of a majority function can be obtained with 4 NAND gates. **(10)**

x	y	z	f	
0	0	0	0	
0	0	1	0	
0	1	0	0	
0	1	1	1	$\leftarrow \bar{x}yz$
1	0	0	0	
1	0	1	1	$\leftarrow x\bar{y}z$
1	1	0	1	$\leftarrow xy\bar{z}$
1	1	1	1	$\leftarrow xyz$

$$f = \bar{x}yz + x\bar{y}z + xy\bar{z} + xyz.$$

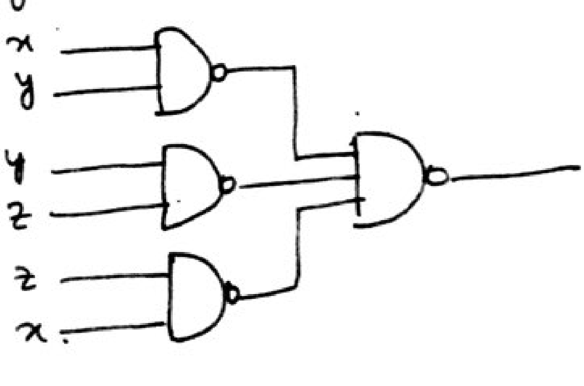
$$= \bar{x}yz + x\bar{y}z + xy(z+\bar{z})$$

$$= \bar{x}yz + x\bar{y}z + xy.$$

$$= \bar{x}yz + x(\bar{y}z + y) = \bar{x}yz + x(y+z)$$

$$= \bar{x}yz + xy + xz = y(\bar{x}z + x) + xz.$$

$$= y(x+z) + xz = xy + yz + zx.$$



6. (a) Find the complete integral of the partial differential equation $(p^2+q^2)x = pz$ and deduce the solution which passes through the curve $x=0, z^2=4y$. **(10)**

$f = (p^2+q^2)x - pz$

$f_x + pf_z = q^2$ & $f_y + qf_z = -pq$

$\therefore \frac{dp}{f_x+pf_z} = \frac{dq}{f_y+qf_z} \Rightarrow \frac{dp}{q^2} = \frac{dq}{-pq} \Rightarrow \frac{dp}{q} = -\frac{dq}{p}.$

$p\,dp + q\,dq = 0 \Rightarrow \underline{p^2+q^2 = a^2}$ (say)

$\therefore a^2x = pz \Rightarrow \boxed{p = a^2x/z}$

$q^2 = a^2 - \frac{a^4x^2}{z^2} = \frac{a^2}{z^2}\left(z^2 - a^2x^2\right)$

$q = \frac{a}{z}\sqrt{z^2-a^2x^2}.$

$\therefore dz = p\,dx + q\,dy = \frac{a^2x}{z}dx + \frac{a}{z}\sqrt{z^2-a^2x^2}\,dy.$

$\Rightarrow \frac{2z\,dz - 2a^2x\,dx}{\sqrt{z^2-a^2x^2}} = 2a\,dy.$

$\Rightarrow \cancel{2}\sqrt{z^2-a^2x^2} = \cancel{2}ay \Rightarrow z^2 - a^2x^2 = (ay+b)^2$

$\Rightarrow \boxed{z^2 = a^2x^2 + (ay+b)^2}$

6. (b) Form a partial differential equation by eliminating a, b and c from the relation $ax^2 + by^2 + cz^2 = 1$. **(08)**

differentiate w.r.t x & y.

$ax + czp = 0$ & $by + czq = 0$. — (1)

differentiating again partially w.r.t x & y

$a + c(p^2 + zr) = 0$ & $b + c(q^2 + zt) = 0$ — (2)

From (1) $\Rightarrow$ $-\frac{a}{c} = \frac{pz}{x}$ & $-\frac{b}{c} = \frac{qz}{y}$

From (2) $\Rightarrow$ $-\frac{a}{c} = p^2 + rz$ & $-\frac{b}{c} = q^2 + tz$.

$\therefore$ $\frac{pz}{x} = p^2 + rz$ & $\frac{qz}{y} = q^2 + tz$.

$$\therefore \left.\begin{aligned} p^2x + rxz - pz = 0 \\ q^2y + tyz - qz = 0 \end{aligned}\right\}.$$

$$\Rightarrow \left.\begin{aligned} &xz\frac{\partial^2 z}{\partial x^2} + x\left(\frac{\partial z}{\partial x}\right)^2 - z\left(\frac{\partial z}{\partial x}\right) = 0 \\ \& \quad &yz\frac{\partial^2 z}{\partial y^2} + y\left(\frac{\partial z}{\partial y}\right)^2 - z\left(\frac{\partial z}{\partial y}\right) = 0. \end{aligned}\right\}$$

6. (c) Reduce $y^2(\partial^2 z/\partial x^2) + x^2(\partial^2 z/\partial y^2) = 0$ to canonical form Let $u = z$ (12)

$y^2 u_{xx} + x^2 u_{yy} = 0$.

$a = y^2$; $b = 0$; $c = x^2$.

charateristics $\Rightarrow$ $\frac{y^2}{2} + i\frac{x^2}{2} = c_1$ & $\frac{y^2}{2} - i\frac{x^2}{2} = c_2$

$\xi = \phi(x,y) = y^2/2$; $\eta = \psi(x,y) = \frac{x^2}{2}$

$\phi_x = 0$; $\phi_y = y$; $\phi_{xx} = 0$; $\phi_{yy} = 1$;

$\psi_x = x$; $\psi_y = 0$; $\psi_{xx} = 1$; $\psi_{yy} = 0$.

Cannonical form $\Rightarrow$

$A u_{\xi\xi} + B u_{\xi\eta} + C u_{\eta\eta} + R = 0$

where $A = y^2 0 + 0 + x^2 y^2 = x^2 y^2$

$B = y^2 0 + 0 + x^2 0 = 0$

$C = y^2 x^2 + 0 + x^2 . 0 = x^2 y^2$.

$R = (y^2 . 0 + 0 + x^2 . 1) u_\xi + (y^2 . 1 + 0 + 0) u_\eta$.

$\therefore$ $x^2 y^2 u_{\xi\xi} + 0 + x^2 y^2 u_{\eta\eta} + x^2 u_\xi + y^2 u_\eta = 0$.

$\Rightarrow$ $2\eta \cdot 2\xi(u_{\xi\xi} + u_{\eta\eta}) + 2\eta u_\xi + 2\xi u_\eta = 0$

$\Rightarrow$ $\boxed{2\xi\eta(u_{\xi\xi} + u_{\eta\eta}) + \eta u_\xi + \xi u_\eta = 0}$

6. (d) The temperature at one end of a bar, 50 cm long with insulated sides, is kept at 0°C and that the other end is kept at 100°C until steady-state condition prevails. The two ends are then suddenly insulated, so that the temperature gradient is zero at each end therafter. Find the temperature distribution. **(20)**

$l = 50$

at $t=0 \Rightarrow$ steady state

$\therefore u = ax + b.$

$x=0;\ u=0 \Rightarrow b=0.$

$x=l;\ u=100 \Rightarrow 100 = a \cdot l \Rightarrow a = \frac{100}{l} = 2.$

$\therefore \underline{u(x,0) = 2x}.$

$\Rightarrow \dfrac{\partial u}{\partial x} = 0$ when $x = 0$ & $x = l$.

The best solution is

$$u(x,t) = (A\cos px + B\sin px)\, e^{-c^2p^2t} \quad \text{—①}$$

$$\frac{\partial u}{\partial x} = p\left(-A\sin px + B\cos px\right) e^{-c^2p^2t} \quad \text{—②}$$

in ② put $x = 0$.

$0 = p(B) \Rightarrow \boxed{B = 0}$

in ② put $x = l$

$0 = p(-A\sin pl) \Rightarrow \sin pl = 0 \Rightarrow pl = n\pi$

$\Rightarrow \boxed{p = n\pi/l}$

$$\therefore u(x,t) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi}{l}x\right) e^{-\frac{n^2\pi^2c^2}{l^2}t} \quad \text{—③}$$

$$\therefore 2x = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi}{l}x\right)$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{l}x\right).$$

$$2x = \frac{A_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{l}x\right) \quad \text{where } a_0 = \frac{A_0}{2}$$

$$\therefore \quad A_0 = \frac{2}{l}\int_0^l (2x)\,dx = \frac{2}{l}\,l^2 = 2l.$$

$$a_n = \frac{2}{l}\int_0^l 2x \cos\left(\frac{n\pi}{l}x\right) dx.$$

$$= \frac{4}{l}\left[x\left(\frac{\sin\left(\frac{n\pi}{l}x\right)}{\frac{n\pi}{l}}\right) - \left(\frac{-\cos\left(\frac{n\pi}{l}x\right)}{n^2\pi^2/l^2}\right)\right]_0^l$$

$$= \frac{4}{l}\left[\frac{l^2}{n^2\pi^2}\cos(n\pi) - \frac{l^2}{n^2\pi^2}\right]$$

$$= \frac{4l}{n^2\pi^2}\left[\cos n\pi - 1\right] = \begin{cases} \frac{-8l}{n^2\pi^2} & ; n \text{ is odd} \\ 0 & ; n \text{ is even.} \end{cases}$$

$$\therefore \quad u(x,t) = l + \sum_{\substack{n=1 \\ n \text{ is odd}}}^{\infty} \left(-\frac{8l}{n^2\pi^2}\right) \cos\left(\frac{n\pi}{l}x\right) e^{-\left(\frac{n^2\pi^2c^2}{l^2}\right)t}$$

$$\Rightarrow \quad u(x,t) = 50 - \frac{400}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^2}\cos\left(\frac{(2m-1)\pi}{50}x\right) \otimes$$

$$\otimes\, e^{-\left(\frac{(2m-1)^2\pi^2c^2}{2500}\right)t}$$

7. (a) Certain corresponding values of x and $\log_{10} x$ are given below:

	x_0	x_1	x_2	x_3
x:	300	304	305	307
$\log_{10}x$:	2.4771	2.4829	2.4843	2.4871
	y_0	y_1	y_2	y_3

Find $\log_{10} 310$ by (i) Lagrange's formula.

(ii) Newton's divided difference formula. **(08)**

$\underline{x = 310}$.

$$y = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f(x_0) + \cdots$$

$$= \frac{6 \cdot 5 \cdot 3}{(-4)(-5)(-7)}(2.4771)$$

$$+ \frac{(10)(5)(3)}{(4)(-1)(-3)}(2.4829)$$

$$+ \frac{(10)(6)(3)}{(5)(1)(-2)}(2.4843) + \frac{(10)(6)(5)}{7 \cdot 3 \cdot 2}(2.4871)$$

$$= -\frac{9}{14}(2.4771) + \frac{25}{2}(2.4829)$$

$$- 18(2.4843) + \frac{50}{7}(2.4871)$$

$$= \frac{436}{175} = \underline{\underline{2.49143}}.$$

7. (b) Evaluate $\int_{-1}^{1} \frac{dx}{1+x^2}$ using Gauss formula for n=2 and n=3. (14)

$f(x) = \frac{1}{1+x^2}$.

<u>n=2</u> ⇒ Two point formula.

$$\int_{-1}^{1} f(x)\,dx = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

$$= \frac{3}{4} + \frac{3}{4}$$

$$= \frac{3}{2} = 1.5 \,//.$$

$\underline{n=3} \Rightarrow$ 3 point formula.

$$\int_{-1}^{1} f(x)\,dx = \frac{1}{9}\left[5f\left(-\sqrt{\frac{3}{5}}\right) + 8f(0) + 5f\left(\sqrt{\frac{3}{5}}\right)\right]$$

$$= \frac{1}{9}\left[5 \cdot \frac{5}{8} + 8 \cdot 1 + 5\ \frac{5}{8}\right]$$

$$= \frac{19}{12} = \underline{\underline{1.58333}}$$

7. (c) Using Runge-Kutta method of fourth order, solve $\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$ with $y(0) = 1$ at $x = 0.2, 0.4$.

(13)

1) $x_0 = 0$; $h = 0.2$; $y_0 = 1$

$$y(x_0+h) = y(0.2) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$k_1 = h f(x_0, y_0) = 0.2 f(0, 1) = 0.2$$

$$k_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.2 f(0.1, 1.1) = 12/61$$

$$k_3 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.2 f(0.1, 1.098) = 0.1967$$

$$k_4 = h f(x_0 + h, y_0 + k_3) = 0.2 f(0.2, 1.1967)$$
$$= 0.18913.$$

7. (b) Evaluate $\int_{-1}^{1} \frac{dx}{1+x^2}$ using Gauss formula for n=2 and n=3. **(14)**

$$f(x) = \frac{1}{1+x^2}.$$

n=2 ⇒ Two point formula.

$$\int_{-1}^{1} f(x)\,dx = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

$$= \frac{3}{4} + \frac{3}{4}$$

$$= \frac{3}{2} = 1.5 \,//.$$

$\underline{n=3} \Rightarrow$ 3 point formula.

$$\int_{-1}^{1} f(x)\,dx = \frac{1}{9}\left[5f\left(-\sqrt{\frac{3}{5}}\right) + 8f(0) + 5f\left(\sqrt{\frac{3}{5}}\right)\right]$$

$$= \frac{1}{9}\left[5 \cdot \frac{5}{8} + 8 \cdot 1 + 5\,\frac{5}{8}\right]$$

$$= \frac{19}{12} = \underline{\underline{1.58333}}$$

7. (c) Using Runge-Kutta method of fourth order, solve $\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$ with $y(0) = 1$ at $x = 0.2, 0.4$. **(13)**

I) $x_0 = 0$; $h = 0.2$; $y_0 = 1$

$$y(x_0+h) = y(0.2) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$k_1 = h f(x_0, y_0) = 0.2 f(0, 1) = 0.2$$

$$k_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.2 f(0.1, 1.1) = 12/61$$

$$k_3 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.2 f(0.1, 1.098) = 0.1967$$

$$k_4 = h f(x_0 + h, y_0 + k_3) = 0.2 f(0.2, 1.1967)$$
$$= 0.18913.$$

$\therefore \underline{y(0\cdot 2) = 1\cdot 196}$

II) $x_0 = 0;\ y_0 = 1\ ;\ h = 0\cdot 4$

$y(x_0 + h) = y(0\cdot 4) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$k_1 = 0\cdot 4\ f(0, 1) = 0\cdot 4$

$k_2 = 0\cdot 4\ f(0\cdot 2, 1\cdot 2) = \frac{14}{37}$

$k_3 = 0\cdot 4\ f(0\cdot 2, 1\cdot 1891) = 0\cdot 378.$

$k_4 = 0\cdot 4\ f(0\cdot 4, 1\cdot 1378) = 0\cdot 33783.$

$\therefore \underline{y(0\cdot 4) = 1\cdot 3751}$

7. (d) Design an algorithm for Trapezoidal rule. **(15)**

Step 1 : Enter step size h.

step 2 : Enter $y_0, y_1, y_2, \ldots, y_n$.

step 3 : sum = 0.

step 4 : for $i = 1$ to $n-1$, step 1

sum = sum + y_i

end for

step 5 : $\text{Integral} = \frac{h}{2}\left[y_0 + y_n + 2\,\text{sum}\right]$

step 6 : Print Integral.