A CONSOLIDATED QUESTION PAPER-CUM-ANSWER BOOKLET

IMS™

(INSTITUTE OF MATHEMATICAL SCIENCES)

TEST SERIES (MAIN)-2013

(QUALITY IMPROVEMENT PROGRAMME)

Test Code: QIP(M) IAS / Test- 11

MATHEMATICS

(PAPER-I)
FULL LENGTH

by **K. VENKANNA**

The person with 14 years of Teaching Experience

Time: Three Hours *Maximum Marks: 250*

INSTRUCTIONS

1. This question paper-cum-answer booklet has 58 pages and has 29 PART/SUBPART questions. Please ensure that the copy of the question paper-cum-answer booklet you have received contains all the questions.
2. Write your Name, Roll Number, Name of the Test Centre and Medium in the appropriate space provided on the right side.
3. A consolidated Question Paper-cum-Answer Booklet, having space below each part/sub part of a question shall be provided to them for writing the answers. Candidates shall be required to attempt answer to the part/sub-part of a question strictly within the pre-defined space. Any attempt outside the pre-defined space shall not be evaluated. "
4. Answer must be written in the medium specified in the admission Certificate issued to you, which must be stated clearly on the right side. No marks will be given for the answers written in a medium other than that specified in the Admission Certificate.
5. Candidates should attempt Question Nos. 1 and 5, which are compulsory, and any **THREE** of the remaining questions selecting at least **ONE** question from each Section.
6. The number of marks carried by each question is indicated at the end of the question. Assume suitable data if considered necessary and indicate the same clearly.
7. Symbols/notations carry their usual meanings, unless otherwise indicated.
8. All questions carry equal marks.
9. All answers must be written in blue/black ink only. Sketch pen, pencil or ink of any other colour should not be used.
10. All rough work should be done in the space provided and scored out finally.
11. The candidate should respect the instructions given by the invigilator.
12. The question paper-cum-answer booklet must be returned in its entirety to the invigilator before leaving the examination hall. Do not remove any page from this booklet.

READ INSTRUCTIONS ON THE LEFT SIDE OF THIS PAGE CAREFULLY

Name: Nitish.K

Roll No.: 149709

Test Centre: Bangalore

Medium: English

Do not write your Roll Number or Name anywhere else in this Question Paper-cum-Answer Booklet.

I have read all the instructions and shall abide by them

Nitish.K

Signature of the Candidate

I have verified the information filled by the candidate above

Signature of the invigilator

IMPORTANT NOTE:

Whenever a question is being attempted, all its parts/ sub-parts must be attempted contiguously. This means that before moving on to the next question to be attempted, candidates must finish attempting all parts/ sub-parts of the previous question attempted. This is to be strictly followed. Pages left blank in the answer-book are to be clearly struck out in ink. Any answers that follow pages left blank may not be given credit.

P.T.O

SECTION - A

Question No. 1

1. (a) Find a non zero vector common to the space spanned by (1, 2, 3), (3, 2, 1) and the space spanned by (1, 0, 1) and (3, 4, 3). **(10)**

$$\begin{bmatrix} 1 & 3 & \vdots & x \\ 2 & 2 & \vdots & y \\ 3 & 1 & \vdots & z \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & \vdots & x \\ 0 & -4 & \vdots & y-2x \\ 0 & -8 & \vdots & z-3x \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & \vdots & x \\ 0 & -4 & \vdots & y-2x \\ 0 & 0 & \vdots & z-2y+x \end{bmatrix} \Rightarrow \underline{x-2y+z=0}$$

$$\begin{bmatrix} 1 & 3 & \vdots & x \\ 0 & 4 & \vdots & y \\ 1 & 3 & \vdots & z \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & \vdots & x \\ 0 & 4 & \vdots & y \\ 0 & 0 & \vdots & z-x \end{bmatrix} \Rightarrow \underline{x-z=0}.$$

$$\Rightarrow \quad x-2y+z=0$$
$$x-z=0$$

$$\begin{bmatrix} 1 & -2 & 1 \\ 1 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -2 \end{bmatrix}$$

$$\Rightarrow \quad x-2y+z=0$$
$$y-z=0.$$

Let $z=1$, $y=1$ $\Rightarrow$ $x-2+1=0 \Rightarrow x=1$

$\Rightarrow$ $[1, 1, 1]$ is a vector common to the two given spaces.

(b) Suppose $T:\mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation whose matrix, with respect to the standard basis is $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 0 \\ 1 & 0 & 1 \end{bmatrix}$.

Find $T^{-1}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right)$. **(10)**

$$T = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow T^{-1} = \frac{1}{4}\begin{bmatrix} -3 & 1 & 6 \\ 1 & 1 & -2 \\ 3 & -1 & -2 \end{bmatrix}$$

$$\Rightarrow T^{-1}(1,2,3) = \frac{1}{4}\begin{bmatrix} -3 & 1 & 6 \\ 1 & 1 & -2 \\ 3 & -1 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 17/4 \\ -3/4 \\ -5/4 \end{bmatrix}$$

P.T.O

(c) Find a and b such that $\underset{x\to 0}{Lt}\dfrac{x(1+a\cos 2x)+b\sin 2x}{x^3}=1.$ **(10)**

$$\lim_{x\to 0}\frac{(1+a\cos 2x)+x(-2a\sin 2x)+2b\cos 2x}{3x^2}\quad \cdots\cdots \left(\frac{1+a+2b}{0}\right)$$

$$\Rightarrow \boxed{1+a+2b=0}. \quad \text{— } (1)$$

$$\lim_{x\to 0}\frac{-2a\sin 2x+(-2a\sin 2x)+x(-4a\cos 2x)-4b\sin 2x}{6x}.$$

$$=\lim_{x\to 0}\frac{-4a\sin 2x-4ax\cos 2x-4b\sin 2x}{6x}\quad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{-4(a+b)\cdot 2\cos 2x - 4a(\cos 2x - 2x\sin 2x)}{6}$$

$$= \frac{-8(a+b) - 4a}{6} = 1$$

$$\Rightarrow 4(a+b) + 2a = -3 \quad \Rightarrow \boxed{6a + 4b = -3.} \text{ --- (2)}$$

$$\Rightarrow \boxed{a = -1/4} \ \& \ \boxed{b = -3/8}$$

(d) If $f(x,y)=\dfrac{x^3y^3}{x^2+y^2},(x,y)\neq(0,0)$ and $f(0,0)=0$, then show that f is continuous at (0, 0). **(10)**

Put $x = r\cos\theta$
$y = r\sin\theta$.

$$|f(x,y)-f(0,0)| = \left|\frac{x^3y^3}{x^2+y^2}\right|$$

$$= \left|\frac{r^6\cos^3\theta\sin^3\theta}{r^2}\right| \leq |r|^4 = (x^2+y^2)^2 < \epsilon$$

if $x^2 < \dfrac{\sqrt{\epsilon}}{2}$ & $y^2 < \dfrac{\sqrt{\epsilon}}{2}$

if $|x| < \sqrt{\dfrac{\sqrt{\epsilon}}{2}}$; $|y| < \sqrt{\dfrac{\sqrt{\epsilon}}{2}}$

let $\sqrt{\dfrac{\sqrt{\epsilon}}{2}} = \delta$.

Then $|f(x,y)-f(0,0)|<\epsilon$ when $|x-0|<\delta$
and $|y-0|<\delta$

$\Rightarrow$ $f(x,y)$ is continuous at $(0,0)$.

(e) Find the equationof the sphere which touches the sphere $x^2+y^2+z^2-x+3y+2z-3=0$ at the point (1, 1, -1) and passes through the origin.

(10)

equation of tangent plane to the given sphere at (1, 1, -1)

$$x+y-z-\frac{1}{2}(x+1)+\frac{3}{2}(y+1)+(z-1)-3=0$$

$$\Rightarrow 2x+2y-2z-x-1+3y+3+z-1-3=0.$$

$$\Rightarrow x+5y-z-2=0 \quad \text{--- (1)}.$$

equation of any sphere passing through the intersection of given sphere & plane (1)

$$x^2+y^2+z^2-x+3y+2z-3+\lambda(x+5y-z-2)=0 \text{ - (3)}$$

since (3) passes through origin

$$-3+\lambda(-2)=0 \Rightarrow \boxed{\lambda=-3/2.}$$

$\therefore$ (3) becomes.

$$2x^2+2y^2+2z^2-2x+6y+4z-6-3(x+5y-z-2)=0$$

$$\Rightarrow \boxed{2x^2+2y^2+2z^2-5x-9y+7z=0}$$

Question No. 2

2. (a) (i) If $P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, then find P^{50}. **(18)**

(ii) Find the dimension of the subspace

$W = \{(x, y, z, w) \in \mathbb{R}^4 \mid x + y + z + w = 0, x + y + 2z = 0, x + 3y = 0\}$.

To find eigen values of $P \Rightarrow |P - \lambda I| = 0$.

$\lambda = 1, 1, 1$ as P is upper triangular matrix.

To find the eigen vector corresponding to $\lambda = 1$.

$(A - I)X = 0 \Rightarrow \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} X = 0$

$\Rightarrow y + z = 0$.

$\cancel{z = 0}$.

$\because P$ is a 3×3 matrix, ~~bect~~

$P^n = aP^2 + bP + cI$.

Also $PX = 1 \cdot X \Rightarrow P^n X = 1^n X$.

$\Rightarrow P^n X = aP^2 X + bPX + cX$

$\Rightarrow 1^n X = a1^2 X + bX + cX$

$\Rightarrow \underline{1 = a + b + c}$.

$\therefore \underline{P^{50} = aP^2 + bP + cI}$

where $a + b + c = 1$

To find the solution space of homogeneous equation

$$x + y + z + w = 0$$

$$x + y + 2z = 0$$

$$x + 3y = 0$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 0 \\ 1 & 3 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 2 & -1 & -1 \end{bmatrix}$$

$$\cancel{\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ & & & \end{bmatrix}} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & -1 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

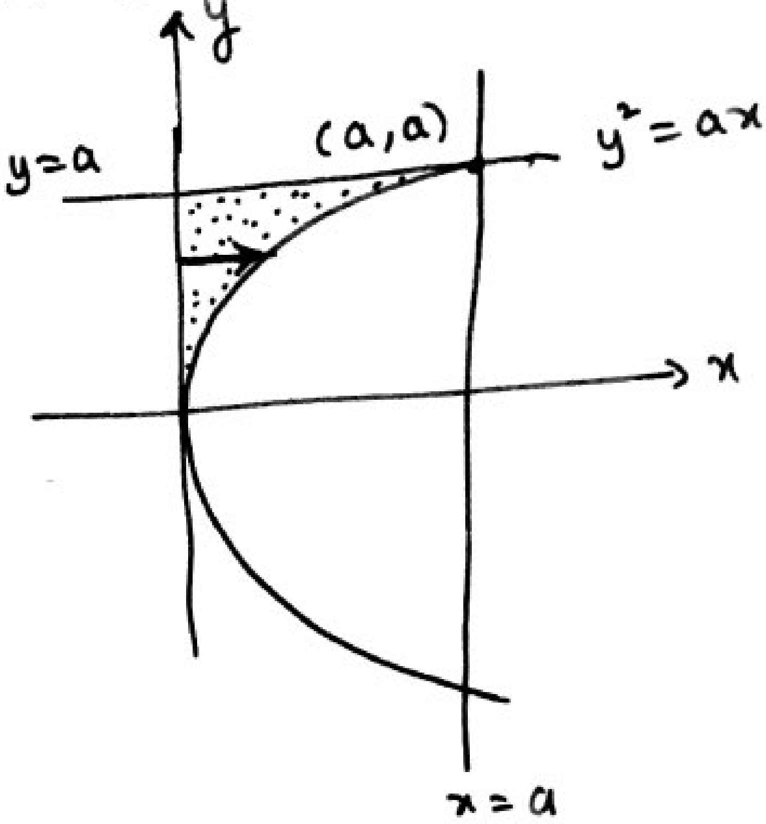
Rank $= 3$.

$$\underline{\dim W = n - r = 4 - 3 = 1.}$$

P.T

(16)

(b) Evaluate the integral $\int_0^a \int_{\sqrt{ax}}^a \frac{y^2\,dx\,dy}{\sqrt{y^4 - a^2x^2}}$ by changing the order of integration.



$$I = \int_{y=0}^{a} \int_{x=0}^{y^2/a} \frac{y^2\,dx}{\sqrt{y^4 - a^2x^2}}\,dy.$$

$$= \frac{1}{a}\int_{y=0}^{a} y^2 \left[\sin^{-1}\left(\frac{ax}{y^2}\right)\right]_{x=0}^{y^2/a} dy.$$

$$= \frac{1}{a}\int_{y=0}^{a} y^2 \left[\sin^{-1} 1 - \sin^{-1}(0)\right] dy$$

$$= \frac{\pi}{2a}\int_{y=0}^{a} y^2\,dy = \frac{\pi}{2a}\,\frac{y^3}{3}\Big]_0^a$$

$$= \frac{\pi}{2a}\,\frac{a^3}{3} = \frac{\pi}{6}a^2 \quad //.$$

(c) Show that the S.D. between any two opposite edges of the tetrahedron formed by the planes $y+z=0, z+x=, x+y=0, x+y+z=a$ is $2a/\sqrt{6}$ and the three lines of S.D. intersect at the point $x=y=z=-a$. **(16)**

one of the edges $\Rightarrow$ $y+z=0$; $z+x=0$.

$$\Rightarrow \frac{x}{1} = \frac{y}{1} = \frac{z}{-1} \quad — ①$$

another edge $\Rightarrow$ $x+y=0$; $x+y+z=a \Rightarrow z=a$.

$$\frac{x}{1} = \frac{y}{-1} = \frac{z-a}{0} \quad — ②.$$

Any point on line ① $\Rightarrow A = (0,0,0)$

Any point on line ② $\Rightarrow B = (0,0,a)$.

Let l, m, n be direction ratios of line of shortest distance $\perp$. solving

$l+m-n=0$; $l-m=0$ $\Rightarrow l=m$;

$$\Rightarrow \frac{l}{1} = \frac{m}{1} = \frac{n}{2}.$$

$\therefore$ Shortest distance = projection of the join of $A(0,0,0)$ & $B(0,0,a)$ on the line having d.c.s l, m, n.

$$= \frac{1(0) + 1(0) + 2(a-0)}{\sqrt{1^2+1^2+2^2}} = \frac{2a}{\sqrt{6}} \, //.$$

equation of any plane through line ① and SD

$$\begin{vmatrix} x & y & z \\ 1 & 1 & -1 \\ 1 & 1 & 2 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} x & y & z \\ 1 & 1 & -1 \\ 0 & 0 & 3 \end{vmatrix} = 0$$

$\Rightarrow 3(x-y) = 0 \quad \Rightarrow \quad x = y$ — ③

equation of any plane through line ② and S.D

$$\begin{vmatrix} x & y & z-a \\ 1 & -1 & 0 \\ 1 & 1 & 2 \end{vmatrix} = 0 \Rightarrow x(-2) - y(2) + (z-a)(2) = 0$$

$\Rightarrow -x - y + z - a = 0 \Rightarrow \underline{x + y - z = -a}$ — ④

eqns ③ & ④ given equation of shortest distance

clearly ③ & ④ satisfy the relation

$\underline{x = y = z = -a.}$

$\therefore$ Line of S.D passes through the point $(-a,-a,-a)$

Similarly, by Symmetry other lines of S.D also pass through the point $(-a,-a,-a)$

$\therefore$ Lines of S.D intersect at the point

$\underline{x = y = z = -a}$

P.T.

Question No. 3

3. (a) (i) Let A be a 3×3 matrix with trace (A) = 3 and det(A) = 2. If 1 is an eigenvalue of A then what are the eigenvalues of the matrix $A^2 - 2I$?

(ii) What is the dimension of the solution space of linear equations $x+y+z+t=0$, $x-y-z-3t=0, -x+y-2z=0$? **(18)**

(c) Show that the locus of the line of intersection of perpendicular tangent planes to the cone $ax^2 + by^2 + cz^2 = 0$ is the cone $a(b+c)x^2 + b(c+a)y^2 + c(a+b)a^2 = 0$. **(16)**

SECTION - B

Question No. 5

5. (a) Solve $\sin x\left(\frac{dy}{dx}\right)+3y=\cos x.$ (10)

$\Rightarrow \frac{dy}{dx} + (3\,\text{cosec}\,x)y = \cot x.$

This is linear in y

$$IF = e^{3\int \text{cosec}\,x\,dy} = e^{3\log\tan(x/2)} = \tan^3(x/2).$$

$\therefore$ Solution is

$$y\cdot IF = \int \cot x\cdot IF\,dx + C$$

$$y\tan^3\left(\frac{x}{2}\right) = \int \cot x\cdot \tan^3\left(\frac{x}{2}\right)dx + C.$$

$$= \int \frac{\cos x}{\sin x}\tan^3(x/2)\,dx + C$$

$$= \int \frac{\cos^2 x/2 - \sin^2 x/2}{2\sin x/2\cos x/2}\,\frac{\sin^3(x/2)}{\cos^3(x/2)}\,dx + C$$

$$= \frac{1}{2}\int \frac{\cos^2 x/2 - \sin^2 x/2}{\cos^4(x/2)}\cdot \sin^2(x/2)\,dx$$

$$= \frac{1}{2}\int \frac{\cos^2\theta - \sin^2\theta}{\cos^4\theta}\sin^2\theta\,d\theta$$

Put: $x/2 = \theta.$

$dx = 2d\theta.$

$$= \int (1-\tan^2\theta)\tan^2\theta\,d\theta = \int(\tan^2\theta - \tan^4\theta)d\theta$$

$$= \int(t^2-t^4)\frac{dt}{1+t^2}.$$

put $\tan\theta = t.$

$$= \int \frac{t^2}{1+t^2}\,dt - \int \frac{t^4}{1+t^2}\,dt$$

$$= \int \frac{t^2+1-1}{t^2+1} - \int \frac{t^4-1+1}{1+t^2}\,dt$$

$$= \int 1 - \frac{1}{t^2+1}\,dt - \left\{\int (t^2-1) + \frac{1}{1+t^2}\,dt\right\}$$

$$= t - \tan^{-1}t - \left[\frac{t^3}{3} - t + \tan^{-1}t\right]$$

$$= 2t - t^3/3 - 2\tan^{-1}t$$

$$= 2\tan\theta - \frac{1}{3}\tan^3\theta - 2\theta = 2\tan\left(\frac{x}{2}\right) - \frac{1}{3}\tan^3\left(\frac{x}{2}\right) - x$$

$$\therefore \boxed{y\tan^3\left(\frac{x}{2}\right) = 2\tan\left(\frac{x}{2}\right) - \frac{1}{3}\tan^{-1}\left(\frac{x}{2}\right) - x + C}$$

(b) Solve $x^2D^2y - 3xDy + 5y = x^2 \sin \log x$. (10)

$$x^2y'' - 3xy' + 5y = x^2 \sin(\log x)$$

Put $\log x = t$ or $x = e^t$.

$$(D_1(D_1-1) - 3D_1 + 5)y = e^{2t}\sin(t) \qquad \text{where } D_1 \equiv \frac{d}{dt}.$$

$$\Rightarrow (D_1^2 - 4D_1 + 5)y = e^{2t}\sin t \qquad \therefore m = 2 \pm i$$

$$\Rightarrow \underline{CF = e^{2t}(c_1\cos t + c_2\sin t)}.$$

P.T.O.

$$PI = \frac{1}{D_1^2 - 4D_1 + 5} e^{2t} \sin t = e^{2t} \frac{1}{(D_1+2)^2 - 4(D_1+2) + 5} \sin t$$

$$= e^{2t} \frac{1}{D_1^2 + 1} \sin t .$$

$$= t e^{2t} \frac{1}{2D_1} \sin t = \frac{t e^{2t}}{2} (-\cos t)$$

$$\therefore y = e^{2t} [c_1 \cos t + c_2 \sin t) - \frac{t}{2} e^{2t} \cos t .$$

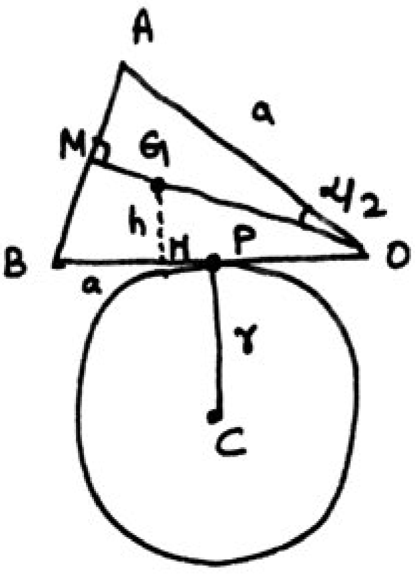
$$\Rightarrow y = x^2 [c_1 \cos(\log x) + c_2 \sin(\log x)]$$
$$- \frac{x^2}{2} \log x \cos(\log x) .$$

(c) A lamina in the form of an isosceles triangle, whose vertical angle is α, is placed on a sphere, of radius r, so that its plane is vertical and one of its equal sides is in contact with the sphere; show that, if the triangle be slightly displaced in its own plane, the equilibrium is stable if $\sin\alpha < 3r/a$, where a is one of the equal sides of the triangle.

(10)

$OM = a\cos\alpha/2$.

$OG = \frac{2}{3}OM = \frac{2}{3}a\cos\alpha/2$



$h = GH = OG\sin\alpha/2$

$= \frac{2}{3}a\cos\alpha/2\ \sin\alpha/2$

$h = \frac{a}{3}\sin\alpha$ = hieght of the Centr of gravity of the upper body above the point of contanct.

ρ_1 = curvature of upper body $= \infty$ ($\because$ plane)

ρ_2 = curvature of the lower body $= r$.

For stable equilibrium.

$\frac{1}{h} > \frac{1}{\rho_1} + \frac{1}{\rho_2} \Rightarrow$

$\Rightarrow \frac{3}{a\sin\alpha} > \frac{1}{r}$

$\Rightarrow a\sin\alpha < 3r \Rightarrow$

P.T.O.

$$\Rightarrow \boxed{\sin\alpha < \frac{3r}{a}}$$

Hence proved.

(d) A particle of mass m is falling under the influence of gravity through a medium whose resistance equals μ times the velocity. If the particle were released from rest, show that the distance fallen through in time t is **(10)**

$$\frac{gm^2}{\mu^2}\left[e^{(-\mu/m)t} - \mathrm{i} + \frac{\mu t}{m}\right].$$

(e) Find the constants a and b so that the surface $ax^2 - byz = (a+2)x$ will be orthogonal to the surface $4x^2y + z^3 = 4$ at the point $(1, -1, 2)$. **(10)**

$\phi_1 \equiv a x^2 - byz - (a+2)x = 0$

$\phi_2 = 4x^2y + z^3 - 4 = 0$.

$\nabla\phi_1 = \hat{i}(2ax - (a+2)) + \hat{j}(-bz) + \hat{k}(-by)$

$\nabla\phi_2 = \hat{i}(8xy) + \hat{j}(4x^2) + \hat{k}(3z^2)$.

at $(1,-1,2)$

$\Rightarrow \nabla\phi_1 = \hat{i}(2a - a - 2) + \hat{j}(-2b) + \hat{k}(b)$

$\nabla\phi_2 = \hat{i}(-8) + \hat{j}(4) + \hat{k}(12)$

$\because$ two surface are othrogonal

$\nabla\phi_1 \cdot \nabla\phi_2 = 0$.

$\Rightarrow -8(a-2) - (2b)\cdot 4 + 12b = 0$

$\Rightarrow -8(a-2) - 8b + 12b = 0 \Rightarrow$

$\Rightarrow -8(a-2) + 4b = 0 \quad \Rightarrow \boxed{2a - b = 4}$

Also $(1,-1,2)$ lies on $\phi_1 \Rightarrow a + 2b = a + 2$.

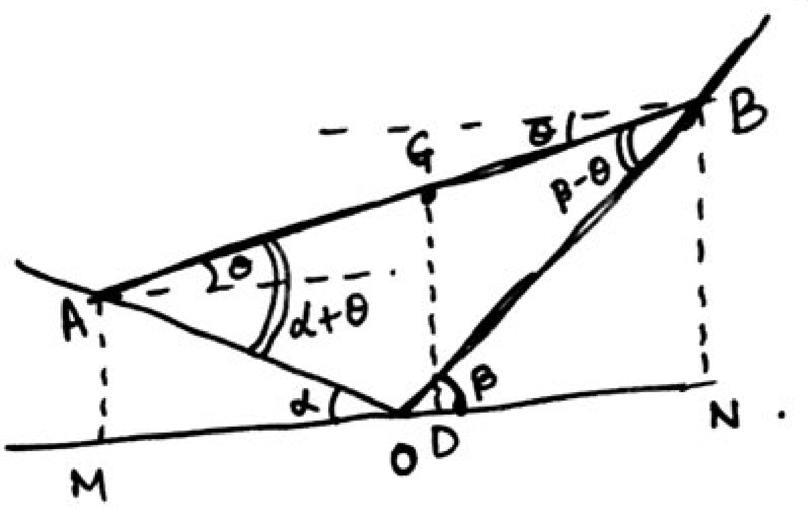
$\Rightarrow \underline{b = 1} \quad \Rightarrow 2a = 5 \quad \Rightarrow a = 5/2$

$\therefore a = 5/2$ and $b = 1$.

Question No. 7

7. (a) A heavy uniform rod, of length 2a, rests with its ends in contact with two smooth inclined planes, of inclination α and β to the horizon. If θ be the inclination of the rod to the horizon, prove, by the principle of virtual work, that $\tan\theta = \frac{1}{2}(\cot\alpha - \cot\beta)$. (12)

Let rod AB make an angle θ with horizontal. Let G be the centre of gravity.



$$GD = \frac{1}{2}(AM + BN)$$

$$= \frac{1}{2}\left(OA \sin\alpha + OB \sin\beta\right)$$

$$\frac{OA}{\sin(\beta-\theta)} = \frac{OB}{\sin(\alpha+\theta)} = \frac{AB}{\sin(\alpha+\beta)} \quad ; AB = 2a.$$

$$GD = \frac{2a}{2}\left[\frac{\sin\alpha \sin(\beta-\theta)}{\sin(\alpha+\beta)} + \frac{\sin\beta \sin(\alpha+\theta)}{\sin(\alpha+\beta)}\right].$$

$$= \frac{a}{\sin(\alpha+\beta)}\left[\begin{array}{l}\sin\alpha\,(\sin\beta\cos\theta - \cos\beta\sin\theta) \\ + \sin\beta\,(\sin\alpha\cos\theta + \cos\alpha\sin\theta)\end{array}\right]$$

$$= \frac{a}{\sin(\alpha+\beta)}\left[\begin{array}{l}\cos\theta\,(\sin\alpha\sin\beta + \sin\beta\sin\alpha) \\ + \sin\theta\,(\sin\beta\cos\alpha - \sin\alpha\cos\beta)\end{array}\right]$$

$$GD = \frac{a}{\sin(\alpha+\beta)}\left[2\cos\theta\sin\alpha\sin\beta + \sin\theta\sin(\beta-\alpha)\right].$$

By virtual work $\Rightarrow -W\,\delta(GD) = 0$.

$$\Rightarrow -2\sin\theta \sin\alpha \sin\beta + \cos\theta \sin(\beta-\alpha) = 0$$

$$2\tan\theta = \frac{\sin(\beta-\alpha)}{\sin\alpha\sin\beta} = \frac{\sin\beta\cos\alpha - \cos\beta\sin\alpha}{\sin\alpha\sin\beta}$$

$$= \cot\alpha - \cot\beta$$

$$\Rightarrow \boxed{\tan\theta = \frac{1}{2}[\cot\alpha - \cot\beta]}$$

(b) A uniform chain of length l, is to be suspended from two points A and B, in the same horizontal line so that either terminal tension is n times that at the lowest point. Show that the span AB must be

$$\frac{1}{\sqrt{(n^2-1)}}\log\left\{n+\sqrt{(n^2-1)}\right\}.$$

(13)

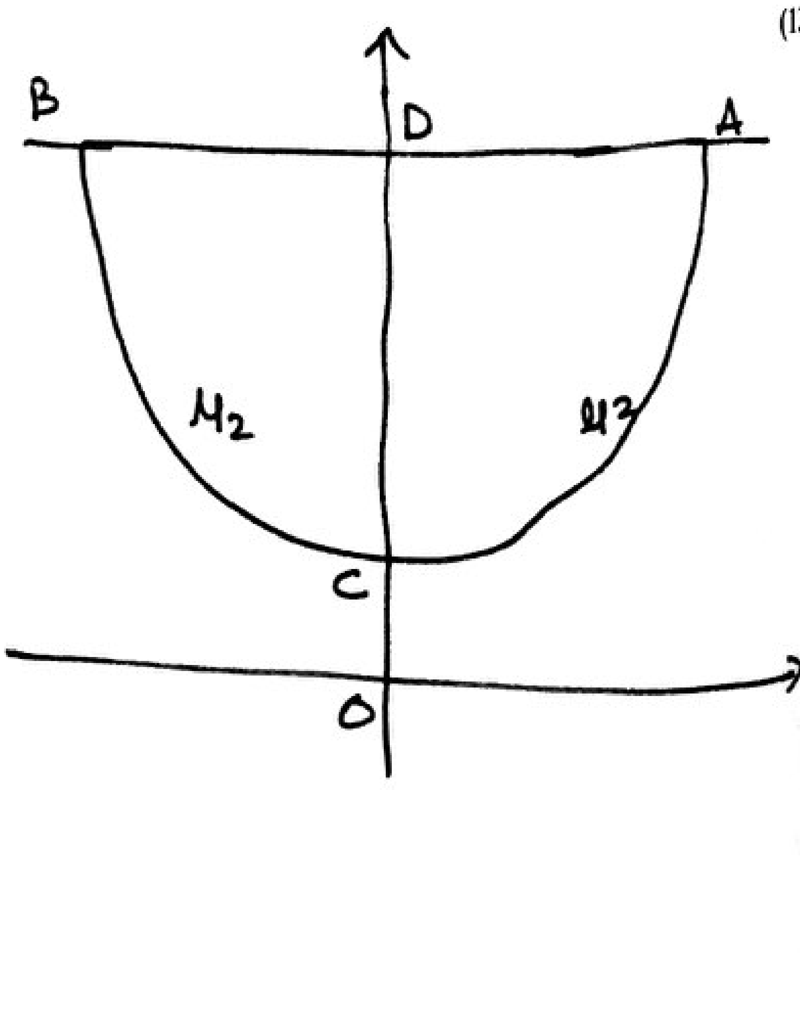
$T_A = nT_c$.

$T_c = wc$

$T_A = wy_A$

$\Rightarrow wy_A = n.wc$

$$\boxed{y_A = nc}.$$



$$y = c\sec\psi \Rightarrow y_A = c\sec\psi_A \Rightarrow nc = c\sec\psi_A$$

$$\Rightarrow \boxed{\sec\psi_A = n}$$

$$y^2 = c^2 + s^2 \Rightarrow y_A^2 = c^2 + s_A^2 \Rightarrow n^2c^2 = c^2 + l^2/4$$

$$\Rightarrow (n^2-1)c^2 = l^2/4 \Rightarrow \boxed{c = \frac{l}{2\sqrt{n^2-1}}}$$

$$\therefore \quad AB = \text{span} = 2x_A$$

$$= 2\cdot c\log(\sec\psi_A + \tan\psi_A)$$

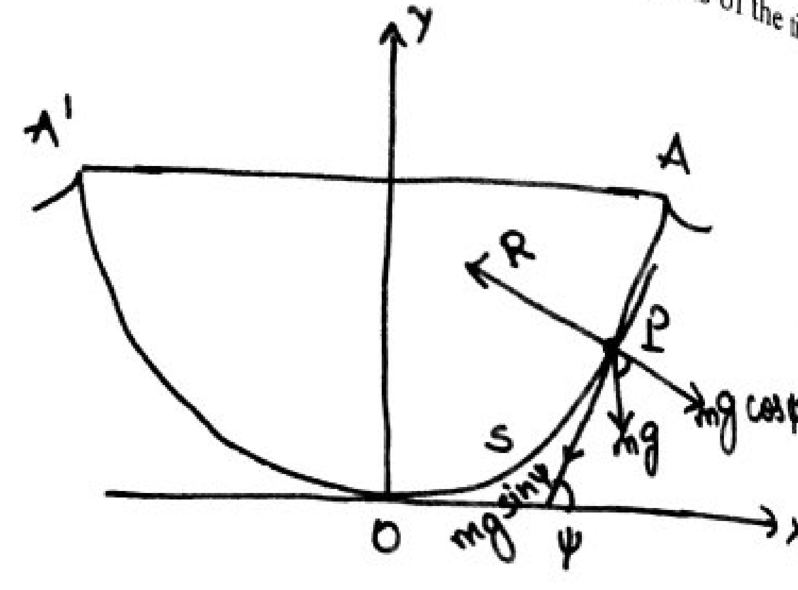
$$\boxed{AB = \frac{l}{\sqrt{n^2-1}}\log\left(n+\sqrt{n^2-1}\right)}$$

$$\because \tan\psi_A = \sqrt{-1+\sec^2\psi_A}$$
$$= \sqrt{n^2-1}.$$

P.T.O.

(c) A particle starts from rest at the cusp of a smooth cycloid whose axis is vertical and vertex downwards. Prove that when it has fallen through half the distance measured along the arc to the vertex, two-thirds of the time of descent will have elapsed. (12)

Let P be the position of the particle at any time t.



$$m\frac{d^2s}{dt^2} = -mg\sin\psi$$

$$\frac{mv^2}{\rho} = R - mg\cos\psi.$$

we know $s = 4a\sin\psi$

$\Rightarrow \dfrac{d^2s}{dt^2} = -\dfrac{g}{4a}s$; multiplying by $2\dfrac{ds}{dt}$ & integrating (1)

$$v^2 = \left(\frac{ds}{dt}\right)^2 = -\frac{g}{4a}s^2 + A.$$

initially at cusp ; $s = 4a$; $v = 0$. $\Rightarrow A = 4ag$.

$$\therefore \left(\frac{ds}{dt}\right)^2 = 4ag - \frac{g}{4a}s^2 = \frac{g}{4a}\left(16a^2 - s^2\right)$$

$$\Rightarrow \frac{ds}{dt} = -\sqrt{\frac{g}{4a}}\sqrt{16a^2 - s^2}$$

$$\Rightarrow dt = -\sqrt{\frac{4a}{g}}\frac{ds}{\sqrt{16a^2 - s^2}}.$$

$$t_1 = -\sqrt{\frac{4a}{g}}\int_{s=4a}^{s=2a}\frac{ds}{\sqrt{16a^2 - s^2}} = \sqrt{\frac{4a}{g}}\cos^{-1}\left(\frac{s}{4a}\right)\Bigg]_{s=4a}^{s=2a}$$

$$t_1 = \sqrt{\frac{4a}{g}}\left[\cos^{-1}\left(\frac{1}{2}\right) - \cos^{-1}(1)\right] \Rightarrow t_1 = \frac{\pi}{3}\sqrt{\frac{4a}{g}}.$$

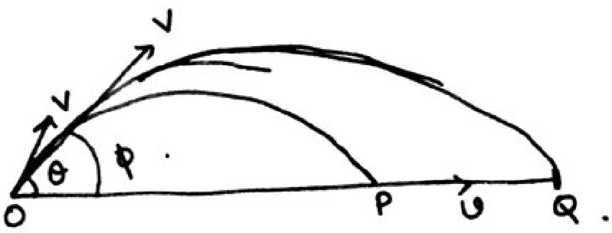
t_d = time of descent $= \frac{1}{4}$ time period $= \frac{1}{4} 2\pi \sqrt{\frac{4a}{g}} = \frac{\pi}{2}\sqrt{\frac{4a}{g}}$

from ①, time period of simple harmonic motion $\ddot{x} = -\mu x$ is $2\pi/\sqrt{\mu}$.

$\Rightarrow \frac{t_1}{t_d} = \frac{\pi/3}{\pi/2} = \frac{2}{3} \Rightarrow \boxed{t_1 = \frac{2}{3} t_d}$

hence the result.

(d) A shot fired with velocity V at an elevation θ strikes a point P on the horizontal plane through the point of projection. If the point P is receding from the gun with velocity v, show that the elevation must be changed to ϕ, where $\sin 2\phi = \sin 2\theta + \frac{2v}{V}\sin\phi.$ (13)



Time taken by the shot to move from OQ = time taken by the point to move from P to Q

$\Rightarrow \frac{2V\sin\phi}{g} = \frac{PQ}{v}$

$PQ = OQ - OP = \frac{V^2 \sin 2\phi}{g} - \frac{V^2 \sin 2\theta}{g}$.

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$$\Rightarrow \frac{2V\sin\phi}{g} = \frac{V^2\sin 2\phi}{vg} - \frac{V^2\sin 2\theta}{vg}$$

$$\Rightarrow 2\sin\phi = \frac{V}{v}\sin 2\phi - \frac{V}{v}\sin 2\theta$$

$$\Rightarrow \frac{2v}{V}\sin\phi = \sin 2\phi - \sin 2\theta$$

$$\Rightarrow \boxed{\sin 2\phi = \sin 2\theta + \frac{2v}{V}\sin\phi}$$

Hence proved.

Question No. 8

8. (a) A particle moves along the curve $x = 4\cos t, y = 4\sin t, z = 6t$. Find the velocity and acceleration at time $t = 0$ and $t = \frac{1}{2}\pi$. Find also the magnitudes of the velocity and acceleration at any time t. **(8)**

$$r = (4\cos t)\,\hat{i} + (4\sin t)\,\hat{j} + (6t)\,\hat{k}$$

$$\dot{r} = (-4\sin t)\,\hat{i} + (4\cos t)\,\hat{j} + 6\,\hat{k} \quad = \text{velocity}$$

$$\ddot{r} = (-4\cos t)\,\hat{i} + (-4\sin t)\,\hat{j} + 0\,\hat{k} = \text{acceleration}$$

at $t=0$

$$\dot{r} = 4\hat{j} + 6\hat{k} \; ; \quad \ddot{r} = -4\hat{i}$$

at $t = \pi/2$

$$\dot{r} = -4\hat{i} + 6\hat{k} \; ; \quad \ddot{r} = -4\hat{j}$$

magnitudes

$$v = |\dot{r}| = \sqrt{16\sin^2 t + 16\cos^2 t + 36} = \sqrt{16+36} = \sqrt{52}$$

$$a = |\ddot{r}| = \sqrt{16\cos^2 t + 16\sin^2 t} = \sqrt{16} = \underline{\underline{4}}\,.$$

(b) If **a** is a constant vector, prove that curl $\frac{\mathbf{a}\times\mathbf{r}}{r^3} = -\frac{\mathbf{a}}{r^3} + \frac{3\mathbf{r}}{r^5}(\mathbf{a}.\mathbf{r})$. (12)

curl $\nabla\times(\phi\vec{A}) = \nabla\phi\times\vec{A} + \phi(\nabla\times\vec{A})$

$\phi = \frac{1}{r^3}$; $\vec{A} = \vec{a}\times\vec{r}$.

$$\text{curl } \frac{\vec{a}\times\vec{r}}{r^3} = -\frac{3}{r^5}\vec{r}\times(\vec{a}\times\vec{r}) + \frac{1}{r^3}\left(\nabla\times(\vec{a}\times\vec{r})\right)$$

$$= -\frac{3}{r^5}\left[(\vec{r}\cdot\vec{r})\vec{a} - (\vec{r}\cdot\vec{a})\vec{r}\right] + \frac{1}{r^3}\sum i\times\frac{\partial}{\partial x}(\vec{a}\times\vec{r})$$

$$= -\frac{3}{r^5}\left(r^2\vec{a} - (\vec{r}\cdot\vec{a})\vec{r}\right) + \frac{1}{r^3}\sum\hat{i}\times\left[\vec{a}\times\frac{\partial\vec{r}}{\partial x}\right]$$

$$= -\frac{3}{r^5}\left(r^2\vec{a} - \vec{r}(\vec{r}\cdot\vec{a})\right) + \frac{1}{r^3}\sum\hat{i}\times(\vec{a}\times\hat{i})$$

$$= -\frac{3}{r^5}\left(r^2\vec{a} - \vec{r}(\vec{r}\cdot\vec{a})\right) + \frac{1}{r^3}\sum\left[(\hat{i}\cdot\hat{i})\vec{a} - (\hat{i}\cdot\vec{a})\hat{i}\right]$$

let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$$= -\frac{3}{r^3}\vec{a} + \frac{3}{r^5}\vec{r}(\vec{r}\cdot\vec{a}) + \frac{1}{r^3}\sum(\vec{a} - a_1\hat{i})$$

$$= -\frac{3}{r^3}\vec{a} + \frac{3}{r^5}\vec{r}(\vec{a}\cdot\vec{r}) + \frac{1}{r^3}(3\vec{a} - \vec{a})$$

$$= -\frac{3}{r^3}\vec{a} + \frac{3}{r^5}\vec{r}(\vec{a}\cdot\vec{r}) + \frac{2\vec{a}}{r^3}$$

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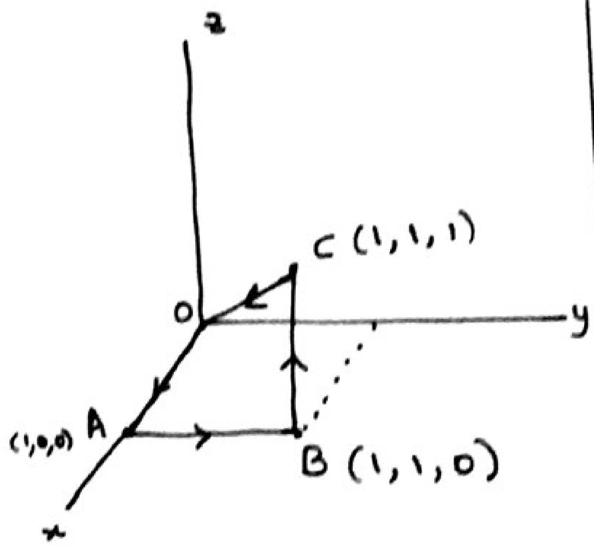
$$= -\frac{\vec{a}}{r^3} + \frac{3\vec{r}}{r^5}(\vec{a}\cdot\vec{r})$$

(c) If $\mathbf{F} = (3x^2+6y)\mathbf{i} - 14yz\mathbf{j} + 20xz^2\mathbf{k}$, evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ where C is the curve consisting of the straight lines from (0, 0, 0) to (1, 0, 0) then to (1,1,0) and then to (1,1,1). **(14)**



$C = OA + AB + BC + CO$

$$\int_C \vec{F}\cdot dr = \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}$$

Along OA $\Rightarrow y=0;\ z=0$.

$$\int_{OA} \vec{F}\cdot dr = \int_{x=0}^{1} 3x^2\,dx = 1$$

Along AB $\Rightarrow x=1;\ dx=0;\ z=0 \Rightarrow$

$$\int_{AB} \vec{F}\,d\vec{r} = 0.$$

Along BC $\Rightarrow x=1,\ y=1 \Rightarrow dx=0,\ dy=0$

$$\int_{BC} \vec{F}\cdot dr = 20\int_{z=0}^{1} z^2\,dz = \frac{20}{3}$$

Along CO $\Rightarrow x=y=z$

$$\int_{CO} \vec{F}\cdot dr = \int_{x=1}^{0}(3x^2+6x - 14x^2+20x^3)\,dx$$

$$= -\frac{13}{3}$$

$$\therefore \int_C \vec{F}\cdot d\vec{r} = 1 + 0 + \frac{20}{3} - \frac{13}{3}$$

$$= \frac{10}{3}\ //$$

P.T.O

[illegible]

[illegible] (10)

Let S_{net} be the entire closed surface consisting of S and S'

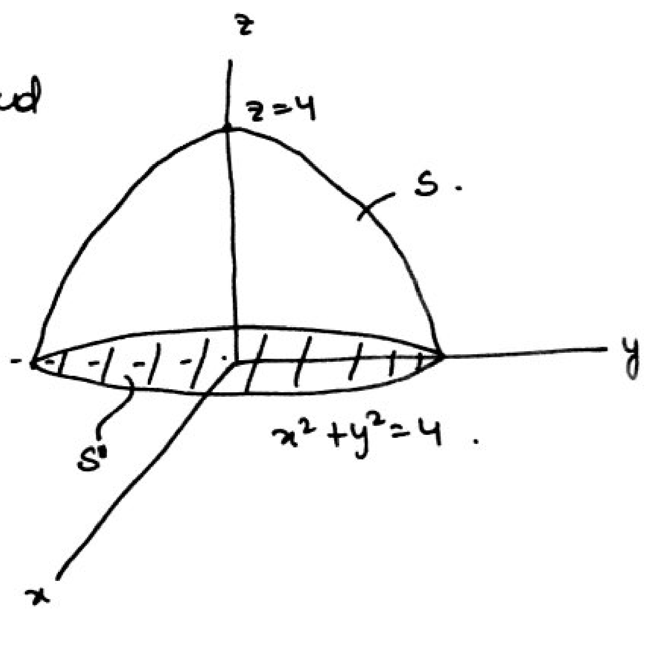
$\iint_{S_{net}} (\nabla \times \vec{F}) \cdot \hat{n} \, dS$

$= \iiint_V \operatorname{div}(\nabla \times \vec{F}) \, dV$

$= 0$ as $\operatorname{div}(\operatorname{curl} \vec{F}) = 0$

(d) Evaluate $\iint_S (\nabla \times F) \cdot \mathbf{n}\, dS$, where $F = (x^2 + y - 4)\mathbf{i} + 3xy\mathbf{j} + (2xz + z^2)\mathbf{k}$ and S is the surface of the paraboloid $z = 4 - (x^2 + y^2)$ above the xy–plane. **(16)**

Let S_{tot} be the entire closed surface consisting of S and S'.



$$\iint_{S_{tot}} (\nabla \times \vec{F}) \cdot \hat{n}\, ds$$

$$= \iiint_V \operatorname{div}(\nabla \times \vec{F})\, dV$$

$= 0$ as $\operatorname{div}(\operatorname{curl} \vec{F}) = 0$.

$$\therefore \iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, dS = -\iint_{S'} (\nabla \times \vec{F}) \cdot \hat{n}\, dS. = I$$

only S' i.e x-y plane $\Rightarrow \hat{n} = -\hat{k}$.

and $z = 0$.

$$I = \iint_{S'} (3y-1)\, dx\, dy.$$

$$= 3\iint_{S'} y\, dx\, dy - \iint_{S'} dx\, dy.$$

$$= 3\int_{x=-2}^{2} \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} y\, dy\, dx - \pi \times 4.$$

$$\underline{\underline{I = -4\pi}}$$