A CONSOLIDATED QUESTION PAPER-CUM-ANSWER BOOKLET

IMS™
(INSTITUTE OF MATHEMATICAL SCIENCES)

TEST SERIES (MAIN)-2013
(QUALITY IMPROVEMENT PROGRAMME)

Test Code: QIP(M) IAS / Test- 10

MATHEMATICS (PAPER-II) FULL LENGTH

by K. VENKANNA

The person with 14 years of Teaching Experience

Time: Three Hours — *Maximum Marks: 250*

INSTRUCTIONS

1. This question paper-cum-answer booklet has 58 pages and has 29 PART/SUBPART questions. Please ensure that the copy of the question paper-cum-answer booklet you have received contains all the questions.
2. Write your Name, Roll Number, Name of the Test Centre and Medium in the appropriate space provided on the right side.
3. A consolidated Question Paper-cum-Answer Booklet, having space below each part/sub part of a question shall be provided to them for writing the answers. Candidates shall be required to attempt answer to the part/sub-part of a question strictly within the pre-defined space. Any attempt outside the pre-defined space shall not be evaluated. "
4. Answer must be written in the medium specified in the admission Certificate issued to you, which must be stated clearly on the right side. No marks will be given for the answers written in a medium other than that specified in the Admission Certificate.
5. Candidates should attempt Question Nos. 1 and 5 which are compulsory, and any **THREE** of the remaining questions selecting at least **ONE** question from each Section.
6. The number of marks carried by each question is indicated at the end of the question. Assume suitable data if considered necessary and indicate the same clearly.
7. Symbols/notations carry their usual meanings, unless otherwise indicated.
8. All questions carry equal marks.
9. All answers must be written in blue/black ink only. Sketch pen, pencil or ink of any other colour should not be used.
10. All rough work should be done in the space provided and scored out finally.
11. The candidate should respect the instructions given by the invigilator.
12. The question paper-cum-answer booklet must be returned in its entirety to the invigilator before leaving the examination hall. Do not remove any page from this booklet.

READ INSTRUCTIONS ON THE LEFT SIDE OF THIS PAGE CAREFULLY

Name Nitish.k

Roll No. 149709

Test Centre

Medium

Do not write your Roll Number or Name anywhere else in this Question Paper-cum-Answer Booklet.

I have read all the instructions and shall abide by them

Nitish.k

Signature of the Candidate

I have verified the information filled by the candidate above

Signature of the invigilator

IMPORTANT NOTE:

Whenever a question is being attempted, all its parts/sub-parts must be attempted contiguously. This means that before moving on to the next question to be attempted, candidates must finish attempting all parts/sub-parts of the previous question attempted. This is to be strictly followed.

Pages left blank in the answer-book are to be clearly struck out in ink. Any answers that follow pages left blank may not be given credit.

P.T.O.

SECTION - A

Question No. 1

(a) Discuss the irreducibility of $f(x)=x^4+1$, over rationals. (10)

$$f(x+1) = (x+1)^4 + 1$$

$$= x^4 + 4x^3 + 6x^2 + 4x + 1 + 1$$

$$f(x+1) = x^4 + 4x^3 + 6x^2 + 4x + 2$$

$2|2,\ 2|4,\ 2|6,\ 2|4$

but $2 \nmid 1$ and $2^2 \nmid 2$.

$\therefore$ By Eisenstien's criteria

$f(x+1)$ is irreducible over Q

$\Rightarrow f(x)$ is irreducible over Q.

Eisenstien Criteria: $f(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_{n-1}x^{n-1} + a_n x^n$

There is a prime number p such that

$p|a_0,\ p|a_1, \dots, p|a_{n-1}$; $p \nmid a_n$, $p^2 \nmid a_0$.

Then $f(x)$ is irreducible over Q.

(b) Examine the convergence of the integral (10)

$$\int_1^2 \frac{dx}{(1+x)\sqrt{2-x}}$$

2 is the only point of infinite discontinuity.

$f(x) = \dfrac{1}{(1+x)\sqrt{2-x}}$; let $\phi(x) = \dfrac{1}{\sqrt{2-x}}$

$\lim\limits_{x \to 2} \dfrac{f(x)}{\phi(x)} = \lim\limits_{x \to 2} \dfrac{1}{1+x} = \dfrac{1}{3}$ (not equal to zero or infinity

$\therefore \int_1^2 f(x)\,dx$ and $\int_1^2 \phi(x)\,dx$ behave identically.

$\int_1^2 \dfrac{1}{(2-x)^{1/2}}\,dx$ is convergent as $n = \frac{1}{2} < 1$

$\therefore \int_1^2 \dfrac{dx}{(1+x)\sqrt{2-x}}$ also converges.

(c) If $|x|<1$, show that $\frac{1}{1-x}\log\frac{1}{1-x}=\sum_{n=1}^{\infty}\left(1+\frac{1}{2}+\frac{1}{3}+.....+\frac{1}{n}\right)x^n$. **(10)**

$$\log(1-x) = -x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\cdots$$

$$-\log(1-x) = x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\cdots$$

$$(1-x)^{-1} = 1+x+x^2+x^3+\cdots$$

$$\therefore \frac{1}{1-x}\log(1-x) = (1+x+x^2+\cdots)\left(x+\frac{x^2}{2}+\frac{x^3}{3}+\cdots\right)$$

$$= x+\left(1+\frac{1}{2}\right)x^2+\left(1+\frac{1}{2}+\frac{1}{3}\right)x^3+\cdots$$

$$= \sum_{n=1}^{\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)x^n.$$

Hence proved.

(10)

(d) Find the Laurent's expansion of $\frac{z^2}{z^4-1}$ is valid for $0<|z-i|<\sqrt{2}$.

$0<|u|<\sqrt{2}$

Let $z-i=u \Rightarrow z=u+i$;

$$f(z)=\frac{z^2}{z^4-1}=\frac{(u+i)^2}{(u+i)^4-1}$$

$$=\frac{z^2}{(z+1)(z-1)(z+i)(z-i)}=$$

$$=\frac{-1/4}{z+1}+\frac{1/4}{z-1}+\frac{i/4}{z+i}+\frac{-i/4}{z-i}$$

$$=-\frac{1}{4}\frac{1}{u+(i+1)}+\frac{1}{4}\frac{1}{u+(i-1)}+\frac{i}{4}\frac{1}{u+2i}-\frac{i}{4}\frac{1}{u}$$

$$=-\frac{1}{4(i+i)}\left[1+\frac{u}{i+1}\right]^{-1}+\frac{1}{4(i-1)}\left[1+\frac{u}{i-1}\right]^{-1}$$

$$+\frac{i}{4\cdot 2i}\left(1+\frac{u}{2i}\right)^{-1}-\frac{i}{4}\frac{1}{u}.$$

$$=\frac{-1}{4(i+1)}\sum_{n=0}^{\infty}(-1)^n\left(\frac{u}{i+i}\right)^n+\frac{1}{4(i-1)}\sum_{n=0}^{\infty}(-1)^n\left(\frac{u}{i-1}\right)^n$$

$$+\frac{1}{8}\sum_{n=0}^{\infty}(-1)^n\left(\frac{u}{2i}\right)^n-\frac{i}{4u}.$$

$$f(z) = \frac{-1}{4(i+1)} \sum_{n=0}^{\infty} (-1)^n \left(\frac{z-i}{i+1}\right)^n + \frac{1}{4(i-1)} \sum_{n=0}^{\infty} (-1)^n \left(\frac{z-i}{i-1}\right)^n$$

$$+ \frac{1}{8} \sum_{n=0}^{\infty} (-1)^n \left(\frac{z-i}{2i}\right)^n - \frac{i}{4(z-i)}.$$

As the expansion involves both positive and negative powers of $(z-i)$, it is Laurent's expansions.

(e) Find the dual of the following L.P.P: (10)

Maximize $\mathbb{Z} = 2x_1 + x_2$

Subject to $x_1 + 5x_2 \le 10, x_1 + 3x_2 \ge 6, 2x_1 + 2x_2 \le 8;$

$x_2 \ge 0$ and x_1 unrestricted in sign.

Bringing to standard primal form.

Max. $z = 2x_1 + x_2$

s.t $x_1 + 5x_2 \le 10$

$-x_1 - 3x_2 \le -6$

$2x_1 + 2x_2 \le 8.$

Let $x_1 = x_3 - x_4$ where $x_3 \ge 0$ & $x_4 \ge 0$.

$x_3 - x_4 + 5x_2 \le 10$ ...... y_1

$-x_3 + x_4 - 3x_2 \le -6$ ...... y_2

$2x_3 - 2x_4 + 2x_2 \le 8$ ...... y_3

$\therefore$ Dual Min $w = 10y_1 - 6y_2 + 8y_3$

s.t $y_1 - y_2 + 2y_3 \ge 2$

$-y_1 + y_2 - 2y_3 \ge -2$

$5y_1 - 3y_2 + 2y_3 \ge 1$

$\therefore$ Dual is

Minimize $w = 10y_1 - 6y_2 + 8y_3$

subject to $y_1 - y_2 + 2y_3 = 2$

$5y_1 - 3y_2 + 2y_3 \ge 1$

where $y_1, y_2, y_3 \ge 0$.

Question No. 2

(a) Find whether the following statements are true or false. Give a proof in case it is true or else give a counter example **(15)**

(i) There may exist a subgroup of order sixteen in a group of order fifty.

(ii) Let G = <a> be a cyclic group of order 35. Then the index

$$[G:\langle a^7\rangle]=5.$$

(iii) $H=\{e,(1\ 2)\ (3\ 4)\}$ is not a normal subgroup of A_4.

(iv) The group $(\mathbb{Z},+)$ is isomorphic to $(\mathbb{Q},+)$.

(i) Let H be a subgroup of G.

$o(H) = 16$; $o(G) = 50$.

by Lagragis theorem $o(H) \mid o(G)$

$\Rightarrow 16 \mid 50$.

but this is impossible.

$\therefore$ This statement is FALSE.

(ii) $G = \langle a \rangle$; $o(a) = 35 = o(G)$.

$(a^7)^5 = a^{35} = e \Rightarrow o(a^7) = 5$.

~~$\left(\frac{G}{\langle a^7\rangle}\right)$~~ $i(\langle a^7\rangle) = o\left(\frac{G}{\langle a^7\rangle}\right) = \frac{o(G)}{o(\langle a^7\rangle)}$

$= \frac{35}{5} = 7$.

$\therefore$ This statement is false.

(iii) $(123) \in A_4$ as it is even permutation

$(123)^{-1} = (1\,3\,2)$

$(12)(34) \in H$

$(1\,2\,3)(12)(3\,4)(123)^{-1}$

$= (1\,2\,3)(1\,2)(3\,4)(1\,3\,2)$

$= (1\,4)(3\,2) \notin H.$

$\therefore$ H is <u>not</u> normal subgroup of A_4

Given statement is <u>True</u>.

(iv) $\phi : Z \to Q$

<u>cannot be onto</u>:

as $\frac{1}{2} \in Q$ doesnot have a preimage in Z

$\therefore$ There cannot be any isomorphism from $\langle Z, + \rangle$ to $\langle Q, + \rangle$

Given statement is False.

(b) Prove that $\prod_{n=1}^{\infty}\left(1-\frac{1}{n^{2/3}}\right)e^{\frac{1}{n^{2/3}}}$ is absolutely convergent.

(c) Evaluate $\int_0^{2\pi} \frac{\cos\theta}{5+4\cos\theta} d\theta$ by contour integration. **(20)**

$$I = \int_0^{2\pi} \frac{\cos\theta}{5+4\cos\theta} d\theta = R\cdot P \text{ of } \int_0^{2\pi} \frac{e^{i\theta}}{5+4\cos\theta} d\theta.$$

Taking the contour an unit circle i.e $|z|=1$

$$z = e^{i\theta}; \quad \cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+z^{-1}}{2}.$$

$$dz = iz\, d\theta. \Rightarrow d\theta = \frac{dz}{iz}.$$

$$I = R\cdot P \text{ of } \int_C \frac{z}{5+2(z+z^{-1})} \frac{dz}{iz}$$

$$= R.P \text{ of } \frac{1}{i} \int_C \frac{z}{2z^2+5z+2} \, dz$$

$$= R.P \text{ of } \frac{1}{i} \int_C f(z) \, dz \quad \text{where } f(z) = \frac{z}{2z^2+5z+2}$$

Poles of $f(z) \Rightarrow 2z^2+5z+2=0$

$$z = -\frac{1}{2}, -2.$$

only $z = -\frac{1}{2}$ lies within C.

$$\text{Residue } (z=-1/2) = \lim_{z \to -1/2} (z+1/2) \frac{z}{2(z+\frac{1}{2})(z+2)}$$

$$= \frac{-1/2}{2(2-1/2)} = -\frac{1}{6}.$$

$$\therefore \int_C f(z)\,dz = 2\pi i \times (-1/6) = -\frac{\pi i}{3}.$$

$$\therefore I = R.P \text{ of } \frac{1}{i} \times \left(-\frac{\pi i}{3}\right) = -\frac{\pi}{3}.$$

$$\therefore \int_0^{2\pi} \frac{\cos\theta}{5+4\cos\theta} \, d\theta = -\frac{\pi}{3} //.$$

SECTION - B

Question No. 5

(a) Find a partial differential equation by eliminating a, b, c, from $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$. **(10)**

differentiate partially w.r.t x

$$\frac{2x}{a^2} + \frac{2z}{c^2}p = 0 \Rightarrow \frac{x}{a^2} = -\frac{z}{c^2}p. \quad — (1)$$

differentiate partially w.r.t y

$$\frac{2y}{b^2} + \frac{2z}{c^2}q = 0 \Rightarrow \frac{y}{b^2} = -\frac{z}{c^2}q \quad — (2)$$

differentiate (1) again w.r.t x partially.

$$\frac{1}{a^2} = -\frac{1}{c^2}\left[p^2 + zr\right] \quad — (3)$$

$$\frac{1}{b^2} = -\frac{1}{c^2}\left[q^2 + zt\right]. \quad — (4)$$

⇒ ~~given pr~~ equation becomes.

Also (1)$\cdot x$ + (2)$\cdot y$ gives.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = -\frac{z}{c^2}(px + qy). \quad — (5)$$

(3)$\cdot x^2$ + (4)$\cdot y^2$ given.

$$\frac{x^2}{a^2} + \frac{y^2}{c^2} = -\frac{1}{c^2}\left[x^2(p^2+zr) + y^2(q^2+zt)\right] \quad — (6)$$

equating (5) and (6).

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$$z(px+qy) = x^2(p^2+zr) + y^2(q^2+zt).$$

$$\Rightarrow z\left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right) = x^2\left[\left(\frac{\partial z}{\partial x}\right)^2 + z\frac{\partial^2 z}{\partial x^2}\right] + y^2\left[\left(\frac{\partial z}{\partial y}\right)^2 + z\frac{\partial^2 z}{\partial y^2}\right]$$

is the required PDE.

(b) Solve $\left(D^2 - DD' - 2D\right)z = \sin(3x+4y) - e^{2x+y}$. **(10)**

Auxiliary eqn $\Rightarrow m^2 - m - 2 = 0 \Rightarrow m = 2,\ m = -1$

$$C.F = \phi_1(y+2x) + \phi_2(y-x)$$

$$(PI)_1 = \frac{1}{(D-2D')(D+D')}\sin(3x+4y).$$

$$= \frac{1}{(3-2\cdot4)(3+4)}\iint \sin v\,dv\,dv$$

$$= \frac{-1}{35}\left[-\sin(3x+4y)\right] = \frac{1}{35}\sin(3x+4y)$$

$$(PI)_2 = -\frac{1}{(D-2D')}\frac{1}{(D+D')}e^{2x+y}.$$

$$= -\frac{1}{D-2D'}\frac{1}{2+1}\int e^v\,dv$$

$$= -\frac{1}{3}\frac{1}{D-2D'}e^{2x+y}.$$

$$= -\frac{1}{3}\frac{x}{1}e^{2x+y}$$

$$= -\frac{x}{3}e^{2x+y}.$$

$$\therefore\ z = \phi_1(y+2x) + \phi_2(y-x)$$

$$+ \frac{1}{35}\sin(3x+4y) - \frac{x}{3}e^{2x+y}. \;//.$$

c) Using fourth order Runge Kutta method, find the approximate solution at $x = 1.2, x = 1.4$ of the initial value problem $y' = xy, y(1) = 2$. **(10)**

I) $x_0 = 1$; $y_0 = 2$; let $h = 0.2$; $y(x_0+h) = y(1.2) = ?$

$$y(x_0+h) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$k_1 = h f(x_0, y_0) = 0.2 f(1, 2) = 0.4$$

$$k_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.2 f(1.1, 2.2) = 0.484$$

$$k_3 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.2 f(1.1, 2.242) = 0.49324$$

$$k_4 = h f(x_0 + h, y_0 + k_3) = 0.2 f(1.2, 2.49324) = 0.5984$$

$$\therefore \underline{y(x_0+h) = y(1.2) = 2.4921}.$$

II) $x_0 = 1$; $y_0 = 2$; $\underline{h = 0.4}$. ; $f(x, y) = xy$.

$$k_1 = 0.4 f(1, 2) = 0.8$$

$$k_2 = 0.4 f(1.2, 2.4) = 1.152$$

$$k_3 = 0.4 f(1.2, 2.576) = 1.23648$$

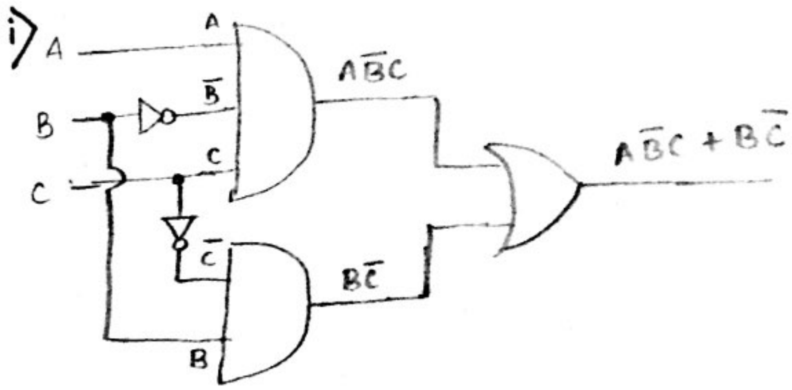
$$k_4 = 0.4 f(1.4, 3.23648) = 1.8124.$$

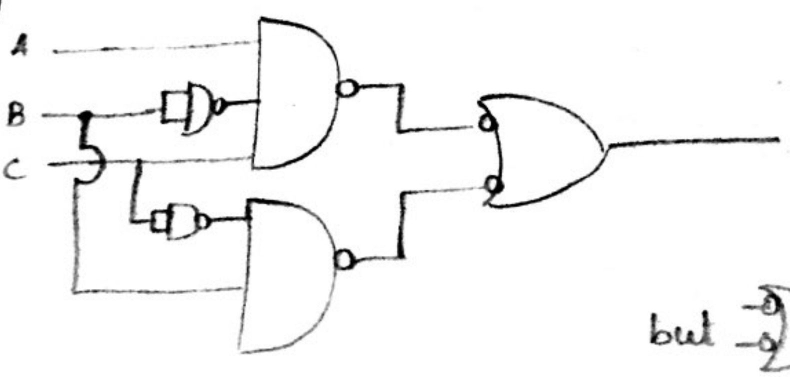
$$\therefore \underline{\underline{y(x_0+h) = y(1.4) = 3.23156}}.$$

(d) (i) Draw the circuit diagram for $\bar{F} = A\bar{B}C + \bar{C}B$ using NAND to NAND logic long. **(10)**

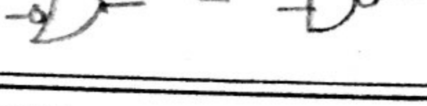
(ii) In a Boolean algebra B, for any a and b prove that $ab' + a'b = 0$ if and only if $a = b$.

i)





but



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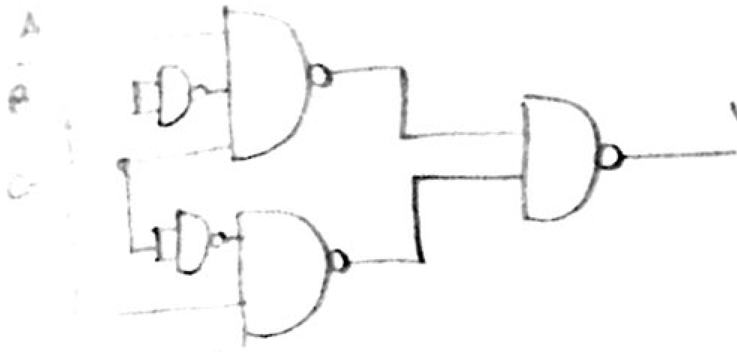

ii) $a\bar{b} + \bar{a}b = 0$

adding 'a' both sides

$\Rightarrow a + a\bar{b} + \bar{a}b = a$

$\Rightarrow a(1+\bar{b}) + \bar{a}b = a$

$\Rightarrow a + \bar{a}b = a$ as $1+\bar{b} = 1$

$\Rightarrow \underline{a + b = a}$. —①

Now adding "b" both sides

$a\bar{b} + \bar{a}b + b = b$

$\Rightarrow a\bar{b} + b(\bar{a}+1) = b$

$\Rightarrow a\bar{b} + b = b$

$\Rightarrow \underline{a+b=b}$ —②

From ① & ② $\boxed{a=b}$ hence proved

Using cylindrical coordinates, write the Hamiltonian and Hamilton's equations for a particle of mass m movi
on the inside of a frictionaless cone $x^2 + y^2 = z^2 \tan^2 \alpha$.

(1

a) Find the integral surface of the linear first order partial differential equation $yp + xq = z - 1$ which passes through the curve $z = x^2 + y^2 + 2, y = 2x$. (10)

$$\frac{dx}{y} = \frac{dy}{x} = \frac{dz}{z-1}$$

$$\Rightarrow \frac{dx}{y} = \frac{dy}{x} \Rightarrow x\,dx = y\,dy$$

$$\Rightarrow x^2 - y^2 = c_1 \quad \text{—} ①.$$

$$\frac{d(x+y)}{x+y} = \frac{dz}{z-1} \Rightarrow \log(x+y) = \log(z-1) + \log c_2 .$$

$$\Rightarrow \frac{x+y}{z-1} = c_2 \quad \text{—} ②.$$

Given curve $z = x^2 + y^2 + 1; \; y = 2x.$

$$x = t; \; y = 2t; \; z = t^2 + 4t^2 + 1 = 5t^2 + 1 .$$

$$x^2 - y^2 = c_1 \Rightarrow t^2 - 4t^2 = c_1 \Rightarrow -3t^2 = c_1 .$$

$$\frac{t+2t}{5t^2} = c_2 \Rightarrow \frac{3}{5t} = c_2 \Rightarrow t = \frac{3}{5c_2}.$$

$$\Rightarrow -3 \cdot \frac{9}{25c_2^2} = c_1 \Rightarrow c_1 c_2^2 = -\frac{27}{25}.$$

$$\Rightarrow (x^2 - y^2)\left(\frac{x+y}{z-1}\right)^2 = -\frac{27}{25}.$$

$$25(x^2-y^2)(x+y)^2+27(z-1)^2=0.$$

is the required surface.

(b) Reduce $\dfrac{\partial^2 z}{\partial x^2}=x^2\left(\dfrac{\partial^2 z}{\partial y^2}\right)$ to canonical form. (15)

$r-x^2t=0. \quad -①$

Comparing with $Rr+Ss+Tt=0$

$R=1;\quad S=0;\quad T=-x^2.$

$S^2-4RT = 4x^2>0$

$\therefore$ Hyperbolic equation

characteristic equation is.

$\lambda^2-x^2=0 \Rightarrow \lambda=\pm x \Rightarrow \lambda_1=x;\ \lambda_2=-x$

$\dfrac{dy}{dx}+\lambda_1=0$ & $\dfrac{dy}{dx}+\lambda_2=0$

$$\frac{dy}{dx} + x = 0; \quad \frac{dy}{dx} - x = 0$$

$$y + \frac{x^2}{2} = c_1 \quad ; \quad y - \frac{x^2}{2} = c_2.$$

choose $u = y + \frac{x^2}{2}$; $v = y - \frac{x^2}{2}$.

$$\partial u/\partial x = x \quad ; \quad \frac{\partial v}{\partial x} = -x$$

$$\partial u/\partial y = 1 \quad ; \quad \frac{\partial v}{\partial y} = 1.$$

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = \left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)x.$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left[x\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\right]$$

$$= \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} + x\left[\frac{\partial}{\partial u}\left(z_u - z_v\right)x + \frac{\partial}{\partial v}(z_u - z_v)(-x)\right]$$

$$= z_u - z_v + x^2\left[z_{uu} - z_{uv} - (z_{uv} - z_{vv})\right]$$

$$r = z_u - z_v + x^2\left[z_{uu} - 2z_{uv} + z_{vv}\right] \quad — ②$$

$$t = \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} + 2\frac{\partial^2 z}{\partial u \partial v} \quad — ③$$

Putting ② & ③ in ①.

$z_u - z_v - 4x^2 z_{uv} = 0$.

$\Rightarrow \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} - 4(u-v)\frac{\partial^2 z}{\partial u \partial v} = 0$

$\therefore \boxed{\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = 4(u-v)\frac{\partial^2 z}{\partial u \partial v}}$

(c) The ends A and B of a rod 20 cm long have the temperatures at 30^0 and 80^0 until steady state prevails. The temperatures of the ends are changed to 40^0 and 60^0 respectively. Find the temperature distribution inthe rod at time t.

(25)

in steady state, $t=0$.

$u = ax + b$.

$30 = b$; $80 = a(20) + 30 \Rightarrow a = 5/2$

$\therefore \boxed{u(x,0) = \frac{5}{2}x + 30}$ initial condition .

for $t > 0$

$u(0,t) = 40$

$u(l,t) = 60$

where $l = 20$

Let $u(x,t) = u_s(x) + u_t(x,t)$

such that $u_s(0) = 40$ & $u_s(l) = 60$.

again $u_s(x) = ax + b$

$\Rightarrow$ using the above condition $a = 1$; $b = 40$

$\therefore$ $u_s(x) = x + 40$.

$u_t(x,t) = u(x,t) - u_s(x)$

$u_t(0,t) = 0$; $u_t(l,t) = 0$ —— ①.

$u_t(x,0) = u(x,0) - u_s(x)$

$= \frac{5}{2}x + 30 - (x+40) = \frac{3}{2}x - 10$ —— ②.

let $u_t(x,t) = (A\cos px + B\sin px)\, e^{-c^2p^2t}$

$\boxed{A = 0}$; $0 = B\sin pl$; as $B \neq 0$

$\Rightarrow$ $\sin pl = 0 \Rightarrow pl = n\pi \Rightarrow \boxed{p = n\pi/l}$

$$\therefore u_t(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{l}x\right) e^{-\frac{n^2\pi^2c^2}{l^2}t}$$

Put $t = 0$

$$\frac{3}{2}x - 10 = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{l}x\right)$$

This is sine half-range fourier series in $(0, l)$

$$b_n = \frac{2}{l}\int_0^l \left(\frac{3}{2}x - 10\right)\sin\left(\frac{n\pi}{l}x\right)dx$$

$$= \frac{2}{l}\left[\left(\frac{3}{2}x-10\right)\left(-\frac{\cos kx}{k}\right) - \left(\frac{3}{2}\right)\left(-\frac{\sin kx}{k^2}\right)\right]_0^l$$

where $k = n\pi/l$. & $l = 20$

$$b_n = \frac{2}{l}\left[\frac{20}{k}\left(-\cos n\pi\right) - \left(-\frac{10}{k}\right)\left(-1\right)\right]$$

$$b_n = -\frac{20}{kl}\left[2\cos n\pi + 1\right]$$

$$= \frac{-20}{n\pi}\left[1 + 2(-1)^n\right]$$

$$\therefore \quad u_t(x,t) = -\frac{20}{\pi}\sum_{n=1}^{\infty}\frac{\left[1+2(-1)^n\right]}{n}\sin\left(\frac{n\pi}{20}x\right)e^{-\frac{n^2\pi^2c^2}{400}t}$$

$$\therefore \quad u(x,t) = x + 40 - \frac{20}{\pi}\sum_{n=1}^{\infty}\frac{\left[1+2(-1)^n\right]}{n}\sin\left(\frac{n\pi}{20}x\right)e^{-\frac{n^2\pi^2c^2}{400}t}$$

Question No. 7

a) Use Newton's method to find the smallest root of the equation $e^x \sin x = 1$ to four places of decimal. (10)

$f(x) = e^x \sin x - 1$; $f'(x) = e^x \sin x + e^x \cos x$

$f(0) = -1 < 0$ & $f(1) = 1.2873 > 0$

Root lies b/w 0 and 1.

let $\underline{x_0 = 0}$

Newton raphson method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \left[\frac{e^{x_n} \sin x_n - 1}{e^{x_n}(\sin x_n + \cos x_n)}\right]$$

$x_0 = 0$

Put $n = 0 \Rightarrow x_1 = 1$

$n = 1 \Rightarrow x_2 = 0.6572$

$x_3 = 0.59118$

$x_4 = 0.58853$

$x_5 = 0.58853$

$x_6 = 0.58853$

$\therefore x = 0.5885$ upto 4 decimal places.

(b) Solve the following equations by Gauss-Seidal Method:

$10x + y + z = 12;\ \ 2x + 10y + z = 13;\ \ 2x + 2y + 10z = 14.$ **(12)**

$$\begin{bmatrix} 10 & 1 & 1 \\ 2 & 10 & 1 \\ 2 & 2 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 13 \\ 14 \end{bmatrix} \Rightarrow AX = B.$$

$A = L + D + U$ where L is lower triangular
D is diagonal
U is upper triangular.

Then Gauss-Seidal method in "matrix form"

$$x^{k+1} = -(D+L)^{-1} U x^{k} + (D+L)^{-1} b$$

$$(D+L) = \begin{bmatrix} 10 & 0 & 0 \\ 2 & 10 & 0 \\ 2 & 2 & 10 \end{bmatrix} ; \quad U = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(D+L)^{-1} = \begin{bmatrix} 0.1 & 0 & 0 \\ -0.02 & 0.1 & 0 \\ -0.016 & -0.02 & 0.1 \end{bmatrix} ; \quad (D+L)^{-1}U = \begin{bmatrix} 0 & 0.1 & 0.1 \\ 0 & -0.02 & 0.08 \\ 0 & -0.016 & 0.036 \end{bmatrix}$$

$$(D+L)^{-1} b = \begin{bmatrix} 1.2 \\ 1.06 \\ 0.948 \end{bmatrix}$$

let $x^{(0)} = [0, 0, 0]$

$$x^{(1)} = \begin{bmatrix} 1.2 \\ 1.06 \\ 0.948 \end{bmatrix} ; \quad x^{(2)} = \begin{bmatrix} 0.9992 \\ 1.0053 \\ 0.999 \end{bmatrix}$$

$$x^{(3)} = \begin{bmatrix} 0.9995 \\ 1.0001 \\ 1 \end{bmatrix} ; \quad x^{(4)} = \begin{bmatrix} 0.9999 \\ 0.9999 \\ 1 \end{bmatrix}$$

$\therefore \quad x = 1$

$y = 1$

$z = 1$

P.T.O.

(c) A reservoir discharging water through sluices at a depth h below the water surface has a surface area A for various values of h as given below: **(13)**

h (ft.)	10	11	12	13	14
A (sq. ft.)	950	1070	1200	1350	1530

If t denotes time in minutes, the rte of fall of the surface is given by $\frac{dh}{dt} = -48\sqrt{h/A}$. Estimate the time taken for the water level to fall from 14 to 10 ft. above the sluices.

	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$
h :	14	13	12	11	10
A :	1530	1350	1200	1070	950
$y(h) = \sqrt{\frac{A}{h}}$:	10.454 y_0	10.2 y_1	10 y_2	9.863 y_3	9.747. y_4

$$t = -\frac{1}{48}\int \sqrt{\frac{A}{h}}\, dh$$

$$= -\frac{1}{48}\int_{h=14}^{h=10} y(h)\, dh.$$

As h is even ; Simpsons $1/3^{rd}$ rule can be applied

$$\int_{h=14}^{10} y(h)\,dt = \frac{k}{3}\left[(y_0 + y_4) + 4(y_1 + y_3) + 2y_2\right]$$

$$= \frac{-1}{3}\left[120.453\right] = -40.151$$

$$\therefore\ t = -\frac{1}{48}(-40.151) = 0.8365 \text{ minutes}$$

or.

$$t. = 50.2 \text{ seconds}.$$

(d) Draw a flow chart for Lagrange's Interpolation formula.