A CONSOLIDATED QUESTION PAPER-CUM-ANSWER BOOKLET

IMS™

(INSTITUTE OF MATHEMATICAL SCIENCES)

TEST SERIES (MAIN)-2014

Test Code: FULL TEST P-II / IAS/T- 06

MATHEMATICS

by K. VENKANNA

The person with 14 years of Teaching Experience

FULL TEST P-II

Time: Three Hours *Maximum Marks: 250*

INSTRUCTIONS

1. This question paper-cum-answer booklet has 51 pages and has 31 PART/SUBPART questions. Please ensure that the copy of the question paper-cum-answer booklet you have received contains all the questions.
2. Write your Name, Roll Number, Name of the Test Centre and Medium in the appropriate space provided on the right side.
3. A consolidated Question Paper-cum-Answer Booklet, having space below each part/sub part of a question shall be provided to them for writing the answers. Candidates shall be required to attempt answer to the part/sub-part of a question strictly within the pre-defined space. Any attempt outside the pre-defined space shall not be evaluated."
4. Answer must be written in the medium specified in the admission Certificate issued to you, which must be stated clearly on the right side. No marks will be given for the answers written in a medium other than that specified in the Admission Certificate.
5. Candidates should attempt Question Nos. 1 and 5, which are compulsory, and any THREE of the remaining questions selecting at least ONE question from each Section.
6. The number of marks carried by each question is indicated at the end of the question. Assume suitable data if considered necessary and indicate the same clearly.
7. Symbols/notations carry their usual meanings, unless otherwise indicated.
8. All questions carry equal marks.
9. All answers must be written in blue/black ink only. Sketch pen, pencil or ink of any other colour should not be used.
10. All rough work should be done in the space provided and scored out finally.
11. The candidate should respect the instructions given by the invigilator.
12. The question paper-cum-answer booklet must be returned in its entirety to the invigilator before leaving the examination hall. Do not remove any page from this booklet.

READ INSTRUCTIONS ON THE LEFT SIDE OF THIS PAGE CAREFULLY

Name: Nitish.K

Roll No.: 149709

Test Centre: Bangalore

Medium:

Do not write your Roll Number or Name anywhere else in this Question Paper-cum-Answer Booklet.

I have read all the instructions and shall abide by them

Nitish.k

Signature of the Candidate

I have verified the information filled by the candidate above

Signature of the invigilator

IMPORTANT NOTE:

Whenever a question is being attempted, all its parts/ sub-parts must be attempted contiguously. This means that before moving on to the next question to be attempted, candidates must finish attempting all parts/ sub-parts of the previous question attempted. This is to be strictly followed. Pages left blank in the answer-book are to be clearly struck out in ink. Any answers that follow pages left blank may not be given credit.

P.T.O.

INDEX TABLE

QUESTION	No.	PAGE NO.	MAX. MARKS	MARKS OBTAINED
1	(a)			
	(b)			
	(c)			
	(d)			
	(e)			
2	(a)			
	(b)			
	(c)			
	(d)			
3	(a)			
	(b)			
	(c)			
	(d)			
4	(a)			
	(b)			
	(c)			
	(d)			
5	(a)			
	(b)			
	(c)			
	(d)			
	(e)			
6	(a)			
	(b)			
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	(d)			
7	(a)			
	(b)			
	(c)			
	(d)			
8	(a)			
	(b)			
	(c)			
	(d)			
Total Marks				

P.T.O.

SECTION-A

1. (a) In S_3 give an example of two elements x, y such that $(x.y)^2 \neq x^2.y^2$. **(10)**

$x = (12)$ & $y = (13)$

$xy = (12)(13) = (132)$

$(xy)^2 = (132)(132) = (123)$

$x^2 = (12)(12) = I$; $y^2 = (13)(13) = I$.

$\therefore (xy)^2 \neq x^2 . y^2$

1. (b) Construct a multiplication table for $Z_2[i]$, the ring of Gaussian integers modulo 2. Is this ring a field? Is it an integral domain? **(10)**

$Z_2[i] = \{a + bi \mid a, b \in Z_2\}$

$= \{0, i, 1, 1+i\}$.

	0	i	1	1+i
0	0	i	1	1+i
i	0	-1	i	-1+i
1	0	i	1	1+i
1+i	0	-1+i	1+i	2i

We know that $Z_2[i] \cong \frac{Z_2[x]}{\langle x^2+1 \rangle}$

as x^2+1 is reducible over Z_2

$\Rightarrow \frac{Z_2[x]}{\langle x^2+1 \rangle}$ is not a field.

Also finite integral domain $\Leftrightarrow$ Field.

$\Rightarrow \frac{Z_2[x]}{\langle x^2+1 \rangle}$ is not a Integral domain

Hence $Z_2[i]$ is neither a Feild nor an Integral Domain.

1. (c) If a function f is continuous in [0, 1], show that

$$\lim_{n \to \infty} \int_0^1 \frac{nf(x)}{1+n^2x^2}\,dx = \frac{\pi}{2} f(0)$$

[10]

$$\int_0^1 \frac{nf(x)}{1+n^2x^2}\,dx = \int_0^{1/\sqrt{n}} \frac{nf(x)}{1+n^2x^2}\,dx + \int_{1/\sqrt{n}}^1 \frac{nf(x)}{1+n^2x^2}\,dx.$$

$$I_1 = \int_0^{1/\sqrt{n}} \frac{nf(x)}{1+n^2x^2}\,dx = f(\xi_1) \int_0^{1/\sqrt{n}} \frac{n\,dx}{1+n^2x^2} \quad \text{where } 0<\xi_1<\frac{1}{\sqrt{n}}.$$

$$= f(\xi_1) \tan^{-1}(nx)\Big]_0^{1/\sqrt{n}} = f(\xi_1)\tan^{-1}\sqrt{n} \to f(0)\,\pi/2$$

as $n \to \infty$.

$$I_2 = \int_{1/\sqrt{n}}^1 \frac{nf(x)}{1+n^2x^2}\,dx = f(\eta) \int_{1/\sqrt{n}}^1 \frac{n\,dx}{1+n^2x^2}\,dx =$$

$$= f(\eta)\left[\tan^{-1} nx\right]_{1/\sqrt{n}}^{1} = f(\eta)\left[\tan^{-1} n - \tan^{-1}\frac{1}{\sqrt{n}}\right]$$

$\rightarrow 0$ as $n \rightarrow \infty$.

$$\therefore \int_0^1 \frac{n f(x)}{1+n^2x^2}\,dx = \lim_{n\to\infty}\int_0^{1/\sqrt{n}} \frac{n f(x)}{1+n^2x^2}\,dx + \lim_{n\to\infty}\int_{1/\sqrt{n}}^{1} \frac{n f(x)}{1+n^2x^2}\,dx$$

$$= \frac{\pi}{2} f(0) + 0$$

$$\therefore \boxed{\int_0^1 \frac{n f(x)}{1+n^2x^2}\,dx = \frac{\pi}{2} f(0)}$$

1. (d) If $f(z) = u + iv$ is an analytic function z, and $u - v = \dfrac{\cos x + \sin x - e^{-y}}{2\cos x - e^{y} - e^{-y}}$: find $f(z)$ subject to the condition $f\left(\dfrac{\pi}{2}\right) = 0$. **(10)**

$$u_x - v_x = \frac{(2\cos x - e^{y} - e^{-y})(-\sin x + \cos x) - (\cos x + \sin x - e^{-y})(-2\sin x)}{(2\cos x - e^{y} - e^{-y})^2}$$

Put $x = z$; $y = 0$

$$u_x - v_x = \frac{(2\cos z - 2)(-\sin z + \cos z) - (\cos z + \sin z - 1)(-2\sin z)}{(2\cos z - 2)^2} \quad \text{--- (i)}$$

$$u_y - v_y = \frac{(2\cos z - 2)(1) - (\cos z + \sin z - 1)(-1+1)}{(2\cos z - 2)^2}$$

$$= \frac{1}{2\cos z - 2}$$

$u_x = v_y$ & $v_x = -u_y$

$\therefore -v_x - u_x = \dfrac{1}{2\cos z - 2}$ — ②

Adding ① & ②

$-2v_x = 0 \Rightarrow v_x = 0 \quad \therefore u_x = \dfrac{1}{2 - 2\cos z}$.

$\therefore f'(z) = u_x + i v_x = \dfrac{1}{2(1-\cos z)} = \dfrac{1}{4\sin^2 z/2}$

$\therefore f(z) = \int \frac{1}{4} \operatorname{cosec}^2 z/2 \, dz = \frac{-1}{2} \cot z/2 + c$.

$f(\pi/2) = 0 \Rightarrow -\frac{1}{2}\cot(\pi/4) + c \Rightarrow \boxed{c = 1/2}$

$\boxed{\therefore f(z) = \frac{1}{2}\left[1 - \cot z/2\right]}$

1. (e) Solve the following assignment problem whose cost matrix is given below. (10)

	I	II	III	IV	V
A	6	5	8	11	16
B	1	13	16	1	13
C	16	11	8	8	8
D	9	14	12	10	16
F	10	13	11	8	16

1	0	3	6	11
0	12	15	0	12
8	3	0	0	0
0	5	3	1	7
2	5	3	0	8

no. of assignments are less than 5.

	I	II	III	IV	V
A	4	[0]	3	9	11
B	[0]	9	12	Ø	9
C	11	3	Ø	3	[0]
D	Ø	2	[0]	1	4
E	2	2	Ø	[0]	5

A → II

B → I

C → V

D → III

E → IV

cost = 5 + 1 + 8 + 12 + 8

= **34**

Alternative solutions are possible.

3. (a) Show that $Z[\sqrt{-6}]$ is not a unique factorization domain. Why does this show that $Z[\sqrt{-6}]$ is not a principal ideal domain? [15]

We shall show that $(1+\sqrt{-6})$ is irreducible but not prime

$1+\sqrt{-6} = (a+\sqrt{-6}\,b)(c+\sqrt{-6}\,d)$. taking conjugate.

$1-\sqrt{-6} = (a-\sqrt{-6}\,b)(c-\sqrt{-6}\,d)$

$\Rightarrow 7 = (a^2+6b^2)(c^2+6d^2)$.

I) $a^2+6b^2 = 1$; $c^2+6d^2 = 7$

II) $a^2+6b^2 = 7$; $c^2+6d^2 = 1$.

~~III) a^2+6b^2~~.

$\Rightarrow$ either $a = \pm 1; b = 0$ or $c = \pm 1, d = 0$

$\Rightarrow$ either $a+\sqrt{-6}\,b$ is a unit or.

$c+\sqrt{-6}\,d$ is a unit

~~$\therefore 1+\sqrt{-6}$ is irreducible~~.

Sol: We shall show that $2+\sqrt{-6}$ is irreducible but not prime. let $2+\sqrt{-6} = (a+b\sqrt{-6})(c+d\sqrt{-6})$

$2-\sqrt{-6} = (a-b\sqrt{-6})(c-d\sqrt{-6})$

$\Rightarrow 10 = (a^2+6b^2)(c^2+6d^2)$.

$\Rightarrow$ either $a+b\sqrt{-6}$ is a unit or $c+d\sqrt{-6}$ is a unit

$\because a^2+6b^2 = 1$ or $c^2+6d^2 = 1$. $\Rightarrow a = \pm 1, b = 0$ or $c = \pm 1, d = 0$

Also $a^2+6b^2 \neq 2$ or 5 & $c^2+6d^2 \neq 2$ or 5 as a, b, c, d are Integers.

$\therefore\ 2+\sqrt{-6}$ is irreducible

Now $(2+\sqrt{-6})(2-\sqrt{-6}) = 10 = 2\cdot 5$.

$\Rightarrow 2+\sqrt{-6} \mid 2\times 5$; but we show that $2+\sqrt{-6}$ does not divide either 2 or 5.

Let $2+\sqrt{-6} \mid 2 \Rightarrow 2 = (2+\sqrt{-6})(a+b\sqrt{-6})$

$\Rightarrow 2 = 2a - 6b + \sqrt{-6}(a+2b)$

$\Rightarrow 2a-6b=2$ & $a+2b=0$ $\Rightarrow a = 2/5,\ b = -1/5$ which is impossible as $a, b \in \mathbb{Z}$.

$\therefore\ 2+\sqrt{-6} \nmid 2$ • ||| ly $2+\sqrt{-6} \nmid 5$

$\therefore\ 2+\sqrt{-6}$ is not prime.

$\therefore\ \mathbb{Z}[\sqrt{-6}]$ is not U.F.D.

We know $\boxed{PID \Rightarrow UFD}$ $\therefore$ if $\mathbb{Z}[\sqrt{-6}]$ is not UFD, it cannot be PID.

3. (b) Examine for term by term integration the series for which $S_n(x) = nxe^{-nx^2}$ indicating the interval over which your conclusion holds. (15)

$S_n(x) = nxe^{-nx^2}$.

$S(x) = \lim_{n\to\infty} S_n(x) = \lim_{n\to\infty} \frac{nx}{e^{nx^2}} = \lim_{n\to\infty} \frac{nx}{1+nx^2+\frac{n^2x^4}{2!}+\cdots}$

$S(x) = 0 \ \forall x$.

In any interval not containing zero say $[a, \infty]$.

$\Rightarrow \int_a^\infty S_n(x)\,dx = \int_a^\infty nxe^{-nx^2} = \frac{1}{2}e^{-na^2}$

$\therefore \lim_{n\to\infty}\int_a^\infty S_n(x)\,dx = \lim_{n\to\infty}\frac{1}{2}e^{-na^2} = 0 = \int_a^\infty S(x)\,dx$

$\therefore$ Term by Term integration holds good in any interval NOT CONTAINING ZERO

P.T.O.

In $[0, \infty]$.

$$\int_0^\infty S_n(x)\,dx = \frac{1}{2}\, e^{-nx^2}\Big]_\infty^0 = \frac{1}{2}$$

$$\lim_{n\to\infty}\int_0^\infty S_n(x)\,dx = \frac{1}{2} \neq \int_0^\infty S(x)\,dx.$$

$\therefore$ Term by Term integration is not valid in any interval containing zero.

Hence.

in $[a, \infty]$; where $a > 0$, Term by Term integration is valid

in $[0, \infty]$, term by term integration is not valid.

3. (c) A company makes three products X, Y and Z out of three raw materials A, B and C. The number of units of raw material requied to produce one unit of the product is as given in the following table: **(20)**

	X	Y	Z
A	1	2	1
B	2	1	4
C	2	5	1

The unit profit contribution of the products X, Y and Z are ₹40, ₹25 and ₹50 respectively. The number of units of raw materials available are 36, 60 and 45 respectively.

(i) Determine the product-mix that will maximize the total profit.

(ii) From the Optimal Simplex table, write the Optimal Solution to its associated dual problem. Also give the economic interpretation.

Let the requirement of product X, Y, Z be x_1, x_2 and x_3 respectively. The problem is to

Max: $z = 40x_1 + 25x_2 + 50x_3$

s.t. $x_1 + 2x_2 + x_3 \le 36$

$2x_1 + x_2 + 4x_3 \le 60$

$2x_1 + 5x_2 + x_3 \le 45$.

s.t $x_1, x_2, x_3 \ge 0$.

Adding slack variables. S_1, S_2 and S_3.

$x_1 + 2x_2 + x_3 + S_1 = 36$

$2x_1 + x_2 + 4x_3 = 60$

$2x_1 + 5x_2 + x_3 = 45$.

	b v	45 x_1	25 x_2	50 x_3	0 S_1	0 S_2	0 S_3	Sol	least ratio
	s_1	1	2	1	1	0	0	36	36/1
	s_2	2	1	(4)	0	1	0	60	15 ←
	s_3	2	5	1	0	0	1	45	45
	z	-45	-25	-50 ↑	0	0	0	-	

	b·v	45	25	50	0	0	0		
0	s_1	1/2	7/4	0	1	-1/4	0	21	42
50	x_3	1/2	1/4	1	0	1/4	0	15	30
0	s_3	(3/2)	19/4	0	0	-1/4	1	30	20 ←
	z	-20 ↑	$-\frac{25}{2}$	0	0	12.5	0	-	

	b·v	45	25	50	0	0	0	Sol.
0	s_1	0	1/6	0	1	-1/6	-1/3	11
50	x_3	0	-4/3	1	0	1/3	-1/3	5
45	x_1	1	19/6	0	0	-1/6	$\frac{2}{3}$	20
	z	0	117.5	0	0	$\frac{55}{6}$	$\frac{40}{3}$	1150

as all objective coefficients are positive, this is the optimal solution.

$x_1 = 20$; $x_2 = 0$; $x_3 = 5$

and $z_{max} = 1150$.

∴ 20 units of product X, 5 units of Product Z and zero units of product Y have to be produced to give maximum profit of 1150 Rs.

optimal solution of the dual.

Min Z_D = max Z = 1150.

$y_1 = 0$; $y_2 = \frac{55}{6}$; $y_3 = \frac{40}{3}$

This implies 0 units of raw material A, $\frac{55}{6}$ units of raw material B & $\frac{40}{3}$ units of raw material C are needed to minimize input cost.

4. (a) Suppose G is a group that exactly eight elements of order 3. How many subgroups of order 3 does G have? **(13)**

SECTION-B

5. (a) Solve $(D^2 - D'^2 - 3D + 3D').z = xy + e^{x+2y}$. **[10]**

$(D-D')(D+D'-3)z = xy + e^{x+2y}$.

$CF = \phi_1(y+x) + e^{3x}\phi_2(y-x)$.

$PI_1 = \frac{1}{D-D'}\frac{1}{D+D'-3}\{xy\}$

$= \frac{1}{-3D}\left[1-\frac{D'}{D}\right]^{-1}\left[1-\frac{D+D'}{3}\right]^{-1}(xy)$

$= -\frac{1}{3D}\left[1+\frac{D'}{D}\right]\left[1+\frac{D+D'}{3}+\frac{1}{9}(2DD')\right](xy)$.

$= -\frac{1}{3D}\left[1+\frac{D'}{D}\right]\left[xy+\frac{1}{3}(x+y)+\frac{2}{9}\right]$

$= -\frac{1}{3D}\left[xy+\frac{1}{3}(x+y)+\frac{2}{9}+\frac{1}{D}\left(x+\frac{1}{3}\right)\right]$

$= -\frac{1}{3}\left[\frac{x^2y}{2}+\frac{1}{3}\left(\frac{x^2}{2}+xy\right)+\frac{2}{9}x+\left(\frac{1}{2}\frac{x^3}{3}+\frac{1}{3}\frac{x^2}{2}\right)\right]$

$= -\frac{1}{3}\left[\frac{x^2y}{2}+\frac{x^2}{3}+\frac{xy}{3}+\frac{2}{9}x+\frac{x^3}{6}\right]$

$PI_2 = \frac{1}{(D-D')(D+D'-3)}e^{x+2y} = -xe^{x+2y}$.

$\therefore z = \phi_1(y+x) + e^{3x}\phi_2(y-x) - \frac{1}{3}\left[\frac{x^2y}{2}+\frac{x^2}{3}+\frac{xy}{3}+\frac{2x}{9}+\frac{x^3}{6}\right] - xe^{x+2y}$

5. (b) By using Newton Raphson method, show that the equation $2e^{-x} = \frac{1}{x+2} + \frac{1}{x+1}$ has two roots greater than -1. Calculate these roots correct to five decimal places. (10)

$f(x) = 2e^{-x} - \left(\frac{1}{x+2} + \frac{1}{x+1}\right)$.

$f(-0.5) > 0$; $f(1) < 0$.

one root lies in $(-0.5, 1)$

$f(-0.8) < 0$; $f(0) > 0$; $f(0.8) < 0$.

Two roots in $(-0.8, 0)$ & $(0, 0.8)$

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}.$$

$$f'(x) = -2e^{-x} + \left(\frac{1}{(x+2)^2} + \frac{1}{(x+1)^2}\right).$$

Taking $x_0 = -0.8$

$x_1 = -0.735$

$x_2 = -0.6982$

$x_3 = -0.6900$

$x_4 = -0.69$.

Taking $x_0 = 0$.

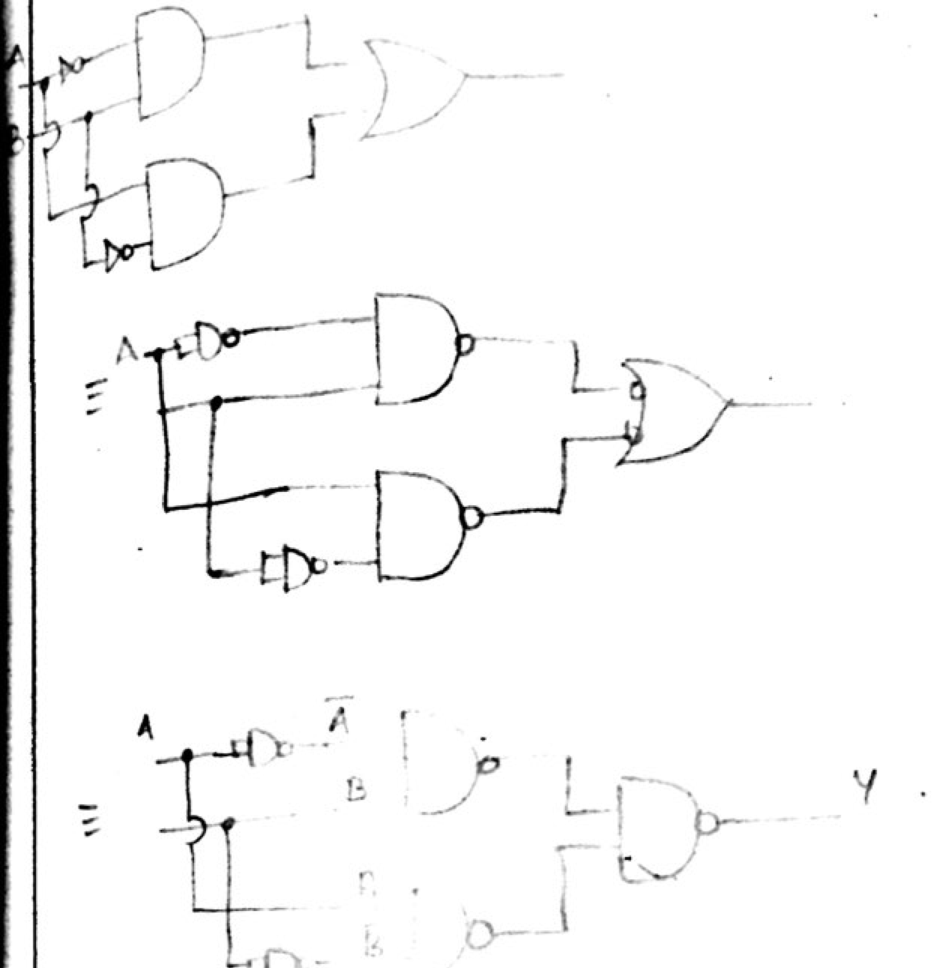
$x_1 = 43$; $x_2 = 0.765$

$x_3 = 0.77$; $x_4 = 0.77$

Two roots are $x = -0.69$ & $x = 0.77$

5. (c) (i) Implement $Y = \bar{A}B + A\bar{B}$ using NAND gates only.

(ii) Find the hexadecimal equivalent of the decimal number $(587632)_{10}$. **[10]**



ii) $(587632)_{10}$ =

16	587632	
16	36727	− 0
16	2295	− 7
16	143	− 7
	8	− F

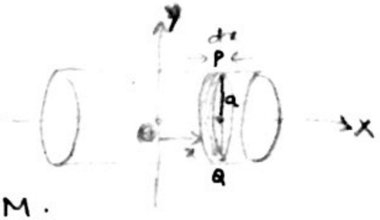
$(587632)_{10} = (8F770)_{16}$

5. (d) Find the M.I. of a right circular cylinder about

(i) its axis,

(ii) a straight line through its C. G. and perpendicular to its axis.

Consider an elementary disc PQ of thickness dx and at a distance x from origin.



elementary mass

$dm = \rho \pi a^2 \delta x$ where $\rho = \frac{M}{\pi a^2 h}$.

$\therefore dI_1 = \frac{1}{2} dm \cdot a^2$

$\therefore I_1 = \frac{1}{2} a^2 \int_{x=-h/2}^{h/2} \rho \pi a^2 \, dx = \frac{a^4 \pi \rho}{2} \cdot 2 \frac{h}{2}$

$I_1 = \frac{a^4 \pi h}{2} \cdot \frac{M}{\pi a^2 h} = \frac{M a^2}{2}$ is the M.I about x-axis i.e axis of cylinder

$$\boxed{I_1 = \frac{1}{2} M a^2}$$

M.I about Y-axis i.e straight line $\perp^r$ to axis and passing through centre of gravity O.

$I_2 = \rho \pi a^2 \int_{x=-h/2}^{h/2} \left(\frac{a^2}{4} + x^2\right) dx$

$= 2 \rho \pi a^2 \left[\frac{a^2}{4} \cdot \frac{h}{2} + \frac{1}{3} \frac{h^3}{8}\right]$

$$\boxed{I_2 = \frac{M}{4}\left[a^2 + \frac{h^2}{3}\right]}$$

5. (e) Prove that liquid motion is possible when velocity at (x, y, z) is given by

$$u=\frac{3x^2-r^2}{r^5},\quad v=\frac{3xy}{r^5},\quad w=\frac{3xz}{r^5},\text{ where } r^2=x^2+y^2+z^2$$

and the stream lines are the intersection of the surfaces, $(x^2+y^2+z^2)^3=c(y^2+z^2)^2$, by the planes passing through Ox. **(10)**

$\frac{\partial u}{\partial x} =$

6. (a) Show that the differential equation of all cones which have their vertex at the origin is $px + qy = z$. Verify that $yz + zx + xy = 0$ is a surface satisfying the above equation. (10)

Equation of the cone is $ax^2 + by^2 = 2cz$.

differentiate partially w.r.t $x \Rightarrow ax = cp$

III'y $by = cq$

$\therefore a = cp/x$ & $b = cq/y$.

$\therefore \frac{cp}{x} x^2 + \frac{cq}{y} y^2 = 2cz$.

$\Rightarrow \boxed{px + qy = 2z}$

$\Rightarrow$

$yz + zx + xy = 0 \Rightarrow yp + z + xp + y = 0$.

$\Rightarrow p(x+y) = -(y+z)$.

III^{ly} $z + yq + xq + x = 0 \Rightarrow (x+y)q = -(x+z)$.

Putting these values of p & q in $px + qy = z$.

$\Rightarrow x\dfrac{y+z}{x+y} + y\left(\dfrac{x+z}{x+y}\right) = -z$.

$\Rightarrow xy + xz + yx + yz = -z(x+y)$.

$\Rightarrow 2xy + z(x+y) = -z(x+y)$

$\Rightarrow 2xy + 2z(x+y) = 0$

$\Rightarrow xy + xz + yz = 0$. which is true

Therefore $yz + zx + xy$ statifies above equation.

(10)

6. (b) Find a complete integral of $p^2x+q^2y=z$.

$\frac{x}{z}\left(\frac{\partial z}{\partial x}\right)^2 + \frac{y}{z}\left(\frac{\partial z}{\partial y}\right)^2 = 1$

$\left(\frac{\sqrt{x}}{\sqrt{z}}\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\sqrt{y}}{\sqrt{z}}\frac{\partial z}{\partial y}\right)^2 = 1$.

let $\frac{1}{\sqrt{x}}dx = dX \Rightarrow 2\sqrt{x} = X$; III^{ly} $2\sqrt{y} = Y$; $2\sqrt{z} = Z$.

$\Rightarrow \left(\frac{\partial Z}{\partial X}\right)^2 + \left(\frac{\partial Z}{\partial Y}\right)^2 = 1 \Rightarrow P^2 + Q^2 = 1$.

$\Rightarrow Z = aX + bY + c$

where $a^2 + b^2 = 1$.

$$2\sqrt{z} = a \cdot 2\sqrt{x} + \sqrt{1-a^2}\, 2\sqrt{y} + c$$

$$\Rightarrow \boxed{\sqrt{z} = a\sqrt{x} + \sqrt{1-a^2}\sqrt{y} + c'}$$

where $c' = 2c$.

6. (c) Reduce $y^2(\partial^2 z/\partial x^2) - x^2(\partial^2 z/\partial y^2) = 0$ to canonical form. **(12)**

$y^2 u_{xx} - x^2 u_{yy} = 0$ where $\boxed{u = z}$.

$a u_{xx} + b u_{xy} + c u_{zz} = 0$.

$a = y^2$; $b = 0$; $c = -x^2$.

$y^2 \lambda^2 = x^2 \Rightarrow \lambda = \pm \frac{x}{y}$.

$\frac{y^2}{2} + \frac{x^2}{2} = c_1$ & $\frac{y^2}{2} - \frac{x^2}{2} = c_2$ are characteristic equations

let $\xi = \phi(x, y) = \frac{y^2}{2} + \frac{x^2}{2}$; $\eta = \psi(x, y)$

$= \frac{y^2}{2} - \frac{x^2}{2}$

P.T.O.

$\phi_x = x;\ \phi_y = y;\ \phi_{xx} = 1;\ \phi_{yy} = 1$

$\psi_x = -x;\ \psi_y = y;\ \psi_{xx} = -1;\ \psi_{yy} = 1.$

$A = y^2x^2 + 0 - x^2y^2 = 0$

$C = y^2x^2 + 0 - x^2y^2 = 0$

$B = 2y^2(-x^2) + 0 + 2(-x^2)(y^2)$

$B = -4x^2y^2.$

$R = (y^2 - x^2)u_\xi + (-y^2 - x^2)u_\eta.$

$Au_{\xi\xi} + Bu_{\xi\eta} + Cu_{\eta\eta} + R = 0$ is the cannonical form.

$\Rightarrow -4x^2y^2u_{\xi\eta} + (y^2 - x^2)u_\xi - (y^2 + x^2)u_\eta = 0$

$\Rightarrow -2(\xi + \eta)(\xi - \eta)u_{\xi\eta} + 2\eta u_\xi - 2\xi u_\eta = 0$

$\Rightarrow (\xi + \eta)(\xi - \eta)z_{\xi\eta} = \eta z_\xi - \xi z_\eta$

6. (d) A string of length l is initially at rest in its equilibrium position and each of its points is given the velocity $(\partial y/\partial t)_{t=0} = v_0 \sin^3(\pi x/l)$ where $0 < x < l$. Find the displacement function. (18)

$y(0,t) = 0 = y(l,t)$ ⇒ Boundary condition

$\left.\frac{\partial y}{\partial t}\right|_{t=0} = v_0 \sin^3\left(\frac{\pi x}{l}\right)$ } initial

$y(x,0) = 0$. } Conditions.

$y(x,t) = (A\cos px + B\sin px)(C\cos cpt + D\sin cpt)$

Put $x=0$.

$0 = A$

Put $x = l$ ⇒ $B\sin pl = 0$ ⇒ $\sin pl = 0$ ⇒ $pl = n\pi$

⇒ $\boxed{p = n\pi/l}$

Put $t=0 \Rightarrow C=0$

$$\therefore y(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{l}x\right) \sin\left(\frac{n\pi c}{l}t\right) \quad \text{--- (1)}$$

$$\frac{\partial y}{\partial t} = \sum_{n=0}^{\infty} \left(b_n \frac{n\pi c}{l}\right) \sin\left(\frac{n\pi}{l}x\right) \cos\left(\frac{n\pi c}{l}t\right)$$

Put $t=0$

$$V_0 \sin^3\left(\frac{\pi x}{l}\right) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{l}x\right) \quad \text{where } B_n = b_n \frac{n\pi c}{l}.$$

$$\frac{3V_0}{4} \sin\left(\frac{\pi x}{l}\right) - \frac{V_0}{4} \sin\left(\frac{3\pi x}{l}\right)$$

$$= B_1 \sin\left(\frac{\pi x}{l}\right) + B_2 \sin\left(\frac{2\pi x}{l}\right) + B_3 \sin\left(\frac{3\pi x}{l}\right) + \cdots$$

$$\Rightarrow B_1 = \frac{3V_0}{4};\ B_2 = 0;\ B_3 = -\frac{V_0}{4};\ B_n = 0 \ \forall n > 3.$$

$$b_1 = \frac{3v_0}{4} \cdot \frac{l}{\pi c};\quad b_3 = -\frac{v_0}{4} \cdot \frac{l}{3\pi c}.$$

Putting in (1)

$$\therefore y(x,t) = \frac{3 l v_0}{4\pi c} \sin\left(\frac{\pi x}{l}\right) \sin\left(\frac{\pi c}{l}t\right) - \frac{l v_0}{12\pi c} \sin\left(\frac{3\pi}{l}x\right) \sin\left(\frac{3\pi c}{l}t\right)$$

7. (a) Apply Gauss-Seidal iteration method to solve the equations $20x+y-2z=17$; $3x+20y-z=-18$; $2x-3y+20z=25$. (12)

$$AX=B \Rightarrow A=\begin{bmatrix}20 & 1 & -2\\ 3 & 20 & -1\\ 2 & -3 & 20\end{bmatrix};\ B=\begin{bmatrix}17\\ -18\\ 25\end{bmatrix}$$

$A=L+D+U$ where L is lower Δ^{lar} matrix
D is diagonal matrix
U is upper Δ^{lar} matrix.

Gauss-Seidal in matrix form

$$x^{k+1} = -(D+L)^{-1}U x^{k} + (D+L)^{-1}B.$$

$$(D+L)=\begin{bmatrix}20 & 0 & 0\\ 3 & 20 & 0\\ 2 & -3 & 20\end{bmatrix};\ U=\begin{bmatrix}0 & 1 & -2\\ 0 & 0 & -1\\ 0 & 0 & 0\end{bmatrix}$$

$$(D+L)^{-1}=\begin{bmatrix}0.05 & 0 & 0\\ -7.5\times10^{-3} & 0.05 & 0\\ -6.125\times10^{-3} & 7.5\times10^{-3} & 0.05\end{bmatrix}$$

$$(D+L)^{-1}U=\begin{bmatrix}0 & 0.05 & -0.1\\ 0 & -7.5\times10^{-3} & -0.035\\ 0 & -6.125\times10^{-3} & 4.75\times10^{-3}\end{bmatrix}$$

$$(D+L)^{-1}B=\begin{bmatrix}17/20\\ -1.027\\ 1.1608\end{bmatrix}$$

Take $x_0 = [0, 0, 0]^T$.

$x_1 = [0.85, -1.027, 1.1608]$

$x_2 = [1.0174, -0.994, 1.149]$

$x_3 = [1.0146, -0.994, 1.1493]$

$x_4 = [1.0146, -0.994, 1.1493]$

$\therefore \begin{cases} x = 1.0146 \\ y = -0.994 \\ z = 1.1493. \end{cases}$

7. (b) A body is in the form of a solid of revolution. The diameter D in cms of its sections distances x cm. from one end are given below. Estimate the volume of the solid. (10

x:	0	2.5	5.0	7.5	10.0	12.5	15.0
D:	5	5.5	6.0	6.75	6.25	5.5	4.0
	D_0	D_1	D_2	D_3	D_4	D_5	D_6

$V = \frac{\pi}{4} \int_0^{15} D^2 \, dx.$

$n = 6.$

$h = \frac{15 - 0}{6} = 2.5$

$V = \frac{\pi}{3} \cdot \frac{h}{3} \left[(D_0^2 + D_6^2) + 4(D_1^2 + D_3^2 + D_5^2) + 2(D_2^2 + D_4^2) \right]$

By Simpson's $1/3$rd rule.

$V = 170.9\pi$

or

$V = 537.02$ cm^3

7. (c) Apply Runge-Kutta fourth order method to find an approximate value of y when $x = 0.2$ given that $dy/dx = x + y$ and $y = 1$ when $x = 0$. (08)

$f(x, y) = x + y$; $h = 0.2$

$x_0 = 0$; $y_0 = 1$;

$y(x_0 + h) = y(0.2) = y_0 + \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$

$k_1 = hf(x_0, y_0) = 0.2 f(0, 1) = 0.2$

$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.2 f(0.1, 1.1) = 0.24$

$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.2 f(0.1, 1.12) = 0.244$

$k_4 = hf(x_0 + h, y_0 + k_3) = 0.2 f(0.2, 1.244) = 0.2888$.

$\therefore\ y(x_0+h) = y(0.2) = 1 + 0.2428$

$$\boxed{y(0.2) = 1.2428}$$

7. (d) Develop an algorithm for Regula-False method to find a root of $f(x) = 0$ starting with two initial iterates x_0 and x_1 to the root such that $\text{Sign}(f(x_0)) \neq \text{Sign}(f(x_1))$. Take n as the maximum number of iterations allowed and eps be the prescribed error. (20)

1) Enter x_0, x_1, f_0, f_1

2) Enter n

3) Enter eps

4) for $i = 0$ to n, in steps of 1.

5) $x_2 = \dfrac{x_0 f_1 - x_1 f_0}{f_1 - f_0}$.

5) $f_2 = f(x_2)$

6) if $(f_1 f_2 < 0)$

then $x_0 = x_2$.

else

$x_1 = x_2$.

7) if ~~[illegible]~~ $\left| \frac{x_1 - x_0}{x_1} \right| < eps$.

then GOTO step 8.

8 ~~else~~

end for :

8) Print x_2, required solution.

9) STOP.