A CONSOLIDATED QUESTION PAPER-CUM-ANSWER BOOKLET

IMS™

(INSTITUTE OF MATHEMATICAL SCIENCES)

TEST SERIES (MAIN)-2014

Test Code: FULL TEST P-II(M) IAS / T- 08

MATHEMATICS

by K. VENKANNA

The person with 14 years of Teaching Experience

FULL TEST P-II

Time: Three Hours *Maximum Marks: 250*

INSTRUCTIONS

1. This question paper-cum-answer booklet has 52 pages and has 35PART/SUBPART questions. Please ensure that the copy of the question paper-cum-answer booklet you have received contains all the questions
2. Write your Name, Roll Number, Name of the Test Centre and Medium in the appropriate space provided on the right side.
3. A consolidated Question Paper-cum-Answer Booklet, having space below each part/sub part of a question shall be provided to them for writing the answers. Candidates shall be required to attempt answer to the part/sub-part of a question strictly within the pre-defined space. Any attempt outside the pre-defined space shall not be evaluated. "
4. Answer must be written in the medium specified in the admission Certificate issued to you, which must be stated clearly on the right side. No marks will be given for the answers written in a medium other than that specified in the Admission Certificate.
5. Candidates should attempt Question Nos. 1 and 5, which are compulsory, and any THREE of the remaining questions selecting at least ONE question from each Section.
6. The number of marks carried by each question is indicated at the end of the question. Assume suitable data if considered necessary and indicate the same clearly.
7. Symbols/notations carry their usual meanings, unless otherwise indicated.
8. All questions carry equal marks.
9. All answers must be written in blue/black ink only. Sketch pen, pencil or ink of any other colour should not be used.
10. All rough work should be done in the space provided and scored out finally.
11. The candidate should respect the instructions given by the invigilator.
12. The question paper-cum-answer booklet must be returned in its entirety to the invigilator before leaving the examination hall. Do not remove any page from this booklet.

READ INSTRUCTIONS ON THE LEFT SIDE OF THIS PAGE CAREFULLY

Name: Nitish.k

Roll No.: 149709

Test Centre: Bangalore

Medium:

Do not write your Roll Number or Name anywhere else in this Question Paper-cum-Answer Booklet.

I have read all the instructions and shall abide by them

Nitish.k

Signature of the Candidate

I have verified the information filled by the candidate above

Signature of the invigilator

IMPORTANT NOTE:

Whenever a question is being attempted, all its parts/ sub-parts must be attempted contiguously. This means that before moving on to the next question to be attempted, candidates must finish attempting all parts/ sub-parts of the previous question attempted. This is to be strictly followed. Pages left blank in the answer book are to be clearly struck out in ink. Any answers that follow pages left blank may not be given credit.

INDEX TABLE

QUESTION	No.	PAGE NO.	MAX. MARKS	MARKS OBTAINED
1	(a)			
	(b)			
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2	(a)			
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3	(a)			
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			Total Marks	

SECTION-A

1. (a) Suppose that a and b are group elements. If $|b| = 2$ and $bab = a^4$, determine the possibilities for $|a|$. **(10)**

$b^2 = e \Rightarrow b = b^{-1}$

$\therefore a^4 = bab = b^{-1}ab$

$a^8 = b^{-1}ab \cdot b^{-1}ab = b^{-1}a^2b$

$a^{16} = b^{-1}a^4b = b^{-1}(bab)b$

$a^{16} = (b^{-1}b)a(bb) = a$

$\Rightarrow \boxed{a^{15} = e}$

$\Rightarrow o(a) \mid 15$

$\Rightarrow o(a) = 1, 3, 5$ or 15

1. (b) Let R be a ring and let $M_2(R)$ be the ring of 2×2 matrices with entries from R. Explain why these two rings have the same characteristic. **(10)**

Let $\text{char}(R) = n$.

$\Rightarrow na = 0 \ \forall a \in R$ where n is the least positive integer.

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(R)$ where $a, b, c, d \in R$.

$\Rightarrow na = 0$; $nb = 0$; $nc = 0$; $nd = 0$

where again n is the least +ve integer.

$$n \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} na & nb \\ nc & nd \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and n is the least positive integer.

$\therefore \text{char}(M_2(R)) = n$

$$\therefore \boxed{\text{char}(R) = \text{char}(M_2(R))}$$

1. (c) Examine the convergence of $\int_0^\infty \left(\frac{1}{1+x} - e^{-x}\right)\frac{dx}{x}$ (10)

consider $I = \int_0^\infty f(x)\,dx$ where $f(x) = \dfrac{e^x - (1+x)}{(1+x)\,e^x\,x}$

clearly $\lim_{x\to 0} f(x) = 0$ $\therefore$ 0 is not a point of infinite discontinuity

$I = \int_0^1 f(x)\,dx + \int_1^\infty f(x)\,dx$ where first integral is a proper integral.

consider $g(x) = \dfrac{1}{x^2}$.

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \left\{ \frac{e^x - (1+x)}{e^x} \quad \frac{x}{1+x} \right\}$$

$$= \lim_{x\to\infty} \frac{e^x - (1+x)}{e^x} \cdot \lim_{x\to\infty} \frac{1}{1/x + 1}$$

$$= \lim_{x\to\infty} \frac{e^x - 1}{e^x} \cdot \frac{1}{0+1} \qquad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \lim_{x\to\infty} \frac{e^x}{e^x} = 1 \text{ (neither zero nor infinity)}$$

$\therefore \int_1^\infty f(x)\,dx$ & $\int_1^\infty g(x)\,dx$ behave identically.

as $\int_1^\infty g(x)\,dx = \int_1^\infty \dfrac{dx}{x^2}$ converges $\Rightarrow \int_1^\infty f(x)\,dx$ converges

$\therefore$ $I = \int_0^\infty f(x)\,dx$ converges.

1. (d) If $u+v=\dfrac{2\sin 2x}{e^{2y}+e^{-2y}-2\cos 2x}$, and $f(z)=u+iv$ is an analytic function of $z=x+iy$, find $f(z)$ in terms of z. **(10)**

Using Thompson-Milne method $\Rightarrow x = z$ & $y = 0$

$$u_x + v_x = 2\left[\frac{(2-2\cos 2z)(2\cos 2z) - 4\sin^2 2z}{(2-2\cos 2z)^2}\right] \quad \text{— (1)}$$

$$u_y + v_y = 2\left[\frac{-\sin 2z}{(2-2\cos 2z)^2}\right] \times (2-2) = 0. \quad \text{— (2)}$$

by C.R equations $u_x = v_y$ & $v_x = -u_y$

$\therefore$ (2) becomes $\Rightarrow u_x - v_x = 0 \Rightarrow \boxed{u_x = v_x}$

$\therefore$ (1) become $u_x = \dfrac{(1-\cos 2z)(\cos 2z) - \sin^2 2z}{(1-\cos 2z)^2}$

$$u_x = \frac{\cos 2z - (\cos^2 2z + \sin^2 2z)}{(1-\cos 2z)^2} = \frac{\cos 2z - 1}{(1-\cos 2z)^2}$$

$$\therefore u_x = \frac{-1}{1-\cos 2z} = \frac{-1}{2\sin^2 z} = -\frac{1}{2}\operatorname{cosec}^2 z.$$

$$\therefore f'(z) = u_x + iv_x = (1+i)u_x = -\frac{1}{2}(1+i)\operatorname{cosec}^2 z$$

$$\Rightarrow f(z) = -\frac{1}{2}(1+i)\int \operatorname{cosec}^2 z \, dz + c$$

$$\boxed{f(z) = \frac{1}{2}(1+i)\cot z + c}$$

1. (e) There are five pumps available for developing five wells. The efficiency of each pump in producing the maximum yield in each well is shown in the table below. In what way should the pumps be assigned so as to maximise the overall efficiency?

Efficiency Well

Pump		W_1	W_2	W_3	W_4	W_5
	P_1	45	40	65	30	55
	P_2	50	30	25	60	30
	P_3	25	20	15	20	40
	P_4	35	25	30	25	20
	P_5	80	60	60	70	50

This is a maximization problem.

Subract each element by the largest element i.e 80

(10)

35	40	15	50	25
30	50	55	20	50
55	60	65	60	40
45	55	50	55	60
0	20	20	10	30

20	25	0	35	10
10	30	35	0	30
15	20	25	20	0
0	10	5	10	15
0	20	20	10	30

	W_1	W_2	W_3	W_4	W_5
P_1	20	15	$\boxed{0}$	35	10
P_2	10	20	35	$\boxed{0}$	30
P_3	15	10	25	20	$\boxed{0}$
P_4	$\cancel{0}$	$\boxed{0}$	5	10	15
P_5	$\boxed{0}$	10	20	10	30

number of assignments = 5 = no. of rows

Therefor the optimal assignment is

$P_1 \longrightarrow W_3$

$P_2 \longrightarrow W_4$

$P_3 \longrightarrow W_5$

$P_4 \longrightarrow W_2$

$P_5 \longrightarrow W_1$

maximum efficientcy.

$= 65 + 60 + 40 + 25 + 80$

$= \underline{\underline{270}}$

4. (a) Find an integer n that shows that the rings Z_n need not have the following properties that the ring of integers has.

(i) $a^2 = a$ implies $a = 0$ or $a = 1$.

(ii) $ab = 0$ implies $a = 0$ or $b = 0$.

(iii) $ab = ac$ and $a \neq 0$ imply $b = c$.

Is the n you found prime? **(12)**

Let $n = 6$ so that we are in the ring Z_6.

① $3^2 = 9 = 3 \Rightarrow 3^2 = 3$

but $3 \neq 0$ not $3 \neq 1$

② $3 \cdot 2 = 6 = 0 \Rightarrow 3 \neq 0$ or $2 \neq 0$

③ $3 \cdot 5 = 15 = 3$ $\Rightarrow$ $3 \cdot 5 = 3 \cdot 3$.

$3 \cdot 3 = 9 = 3$ but $5 \neq 3$.

No n is not a prime.

if n is prime then Z_n is a field and all the above conditions would be satisfied.

4. (b) Let $f : \mathbf{R} \to \mathbf{R}$ be defined by

$$f(x) = \begin{cases} x^2 \sin \dfrac{1}{x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Show that f is differentiable on $\mathbf{R}$ but f' is not continuous on $\mathbf{R}$. (13)

$$Rf'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h\to 0} h \sin\left(\frac{1}{h^2}\right) = 0 \times k = 0$$

where k oscillates finitely b/w -1 & 1

$$Lf'(0) = \lim_{h\to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h\to 0} -h \sin\left(\frac{1}{h^2}\right)$$

$$= 0 \times k = 0 .$$

$\therefore$ f is differentiable at $x = 0$. — ①

for $x \neq 0 \Rightarrow f'(x) = 2x \sin\left(\frac{1}{x^2}\right) - \frac{2}{x} \cos\left(\frac{1}{x^2}\right)$

$\therefore$ f is differentiable $\forall x \neq 0$. — ②

from ① & ② $\Rightarrow$ f is differentiable on R.

$$f'(x) = \begin{cases} 2x \sin\left(\frac{1}{x^2}\right) - \frac{2}{x} \cos\left(\frac{1}{x^2}\right) & ; x \neq 0 \\ 0 & ; x = 0. \end{cases}$$

$\lim_{x \to 0} f'(x)$ does not exists as $\frac{1}{x} \cos\left(\frac{1}{x^2}\right) \to \infty$

$\therefore$ f' is not continuous at orgin and has discontinuity of second kind.

4. (c) Expand $f(z) = \dfrac{z+3}{z(z^2 - z - 2)}$ in powers of z; where (i) $|z| < 1$, (ii) $1 < |z| < 2$, (iii) $|z| > 2$. **(12)**

$$f(z) = -\frac{3}{2z} + \frac{5}{6}\,\frac{1}{z-2} + \frac{2}{3}\,\frac{1}{z+1}$$

① $|z| < 1$

$$f(z) = \frac{-3}{2z} - \frac{5}{12}\left[1 - \frac{z}{2}\right]^{-1} + \frac{2}{3}\left[1 + z\right]^{-1}$$

$$= \frac{-3}{2z} - \frac{5}{12}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^{n} + \frac{2}{3}\sum_{n=0}^{\infty}(-1)^n z^n$$

② $1 < |z| < 2 \Rightarrow \dfrac{1}{|z|} < 1$ & $\dfrac{|z|}{2} < 1$

$$f(z) = -\frac{3}{2z} - \frac{5}{12}\left[1 - \frac{z}{2}\right]^{-1} + \frac{2}{3z}\left[1 + \frac{1}{z}\right]^{-1}$$

$$f(z) = -\frac{3}{2z} - \frac{5}{12}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^{n} + \frac{2}{3z}\sum_{n=0}^{\infty}(-1)^n \frac{1}{z^n}.$$

③ $|z| > 2 \Rightarrow \dfrac{2}{|z|} < 1$

$$\therefore f(z) = -\frac{3}{2z} + \frac{5}{6z}\left[1 - \frac{2}{z}\right]^{-1} + \frac{2}{3z}\left[1 + \frac{1}{z}\right]^{-1}$$

$$= -\frac{3}{2z} + \frac{5}{6z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^{n} + \frac{2}{3z}\sum_{n=0}^{\infty}(-1)^n \frac{1}{z^n}$$

4. (d) Make a graphical representation of the set of constraints of the following LPP. Find the extreme points of the feasible region. Finally, solve the problem graphically.

Maximise $Z = 2x_1 + x_2$

subject to $x_1 + x_2 \geq 5$

$2x_1 + 3x_2 \leq 20$

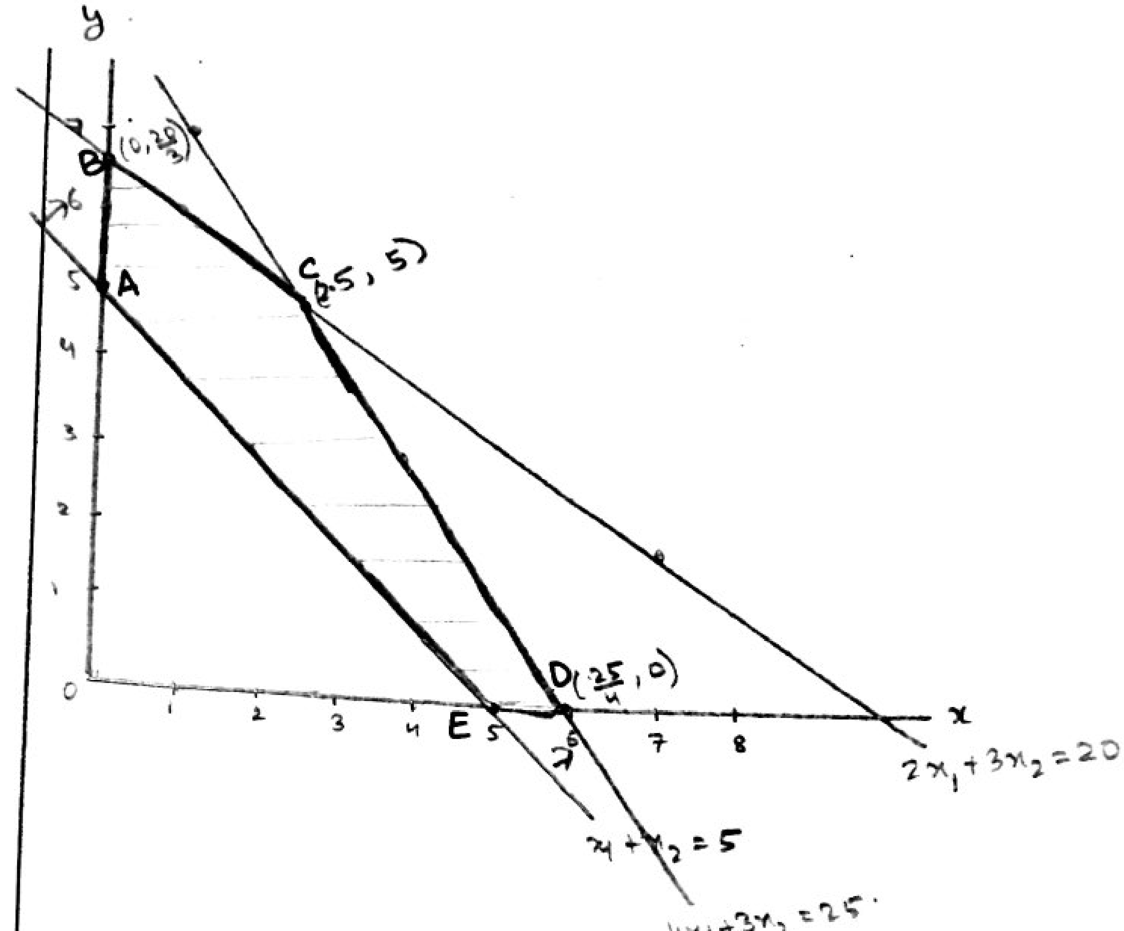
$4x_1 + 3x_2 \leq 25$

$x_1, x_2 \geq 0.$ **(13)**

$x_1 + x_2 = 5 \Rightarrow (0, 5)$ & $(5, 0)$

$2x_1 + 3x_2 = 20 \Rightarrow (1, 6)$ & $(7, 2)$

$4x_1 + 3x_2 = 25 \Rightarrow (1, 7)$ & $(4, 3)$



Extreme point.

$A=(0,5) \Rightarrow z = 5$

$B=(0,\frac{20}{3}) \Rightarrow z = 20/3 = 6.67$

$C=(2.5,5) \Rightarrow z = 10$

$D=(\frac{25}{4},0) \Rightarrow z = 25/2 = 12.5$.

$E=(5,0) \Rightarrow z = 10$.

$\therefore z_{max} = 12.5$

$x_1 = \frac{25}{4}$ & $x_2 = 0$.

SECTION-B

5. (a) Solve $x^2p^2 + y^2q^2 = z^2$. (10)

$$\left(\frac{x}{z}\frac{\partial z}{\partial x}\right)^2 + \left(\frac{y}{z}\frac{\partial z}{\partial y}\right)^2 = 1.$$

let $\frac{1}{x}dx = dX$; $\frac{1}{y}dy = dY$; $\frac{1}{z}dz = dZ$

$\Rightarrow$ $\log x = X$; $\log y = Y$; $\log z = Z$.

$$\therefore \left(\frac{\partial Z}{\partial X}\right)^2 + \left(\frac{\partial Z}{\partial Y}\right)^2 = 1$$

$\Rightarrow$ $P^2 + Q^2 = 1$ where $P = \frac{\partial Z}{\partial X}$ & $Q = \frac{\partial Z}{\partial Y}$

$\therefore$ $Z = aX + bY + c$ where $a^2 + b^2 = 1$

$$\boxed{\log z = a\log x + \sqrt{1-a^2}\,\log y + c'}$$

or.

$$\boxed{z = c x^a y^{\sqrt{1-a^2}}}$$ where $c' = \log c$.

5. (b) Find a surface satisfying $r - 2s + t = 6$ and touching the hyperbolic paraboloid $z = xy$ along its section by the plane $y = x$. **(10)**

$$(D^2 - 2DD' + D'^2)z = 6$$

$$\Rightarrow (D - D')^2 z = 6. \qquad m = 1, 1.$$

$$\underline{C.F = \phi_1(y + x) + x\,\phi_2(y + x)}$$

$$PI = \frac{1}{(D - D')^2} \cdot 6 = \frac{1}{D^2}\left[1 - \frac{D'}{D}\right]^{-2} \cdot 6$$

$$= \frac{1}{D^2}\left[1 + 2\frac{D'}{D}\right] 6 = \frac{1}{D^2} \cdot 6 = 6 \cdot \frac{x^2}{2}$$

$$= 3x^2.$$

$$\left.\begin{aligned} \therefore\ z &= \phi_1(y+x) + x\phi_2(y+x) + 3x^2 \\ z &= xy. \end{aligned}\right\} \text{touch along } y=x.$$

$\therefore$ p & q values are equal.

$$\left.\begin{aligned} &\phi_1'(y+x) + x\phi_2'(y+x) + \phi_2(y+x) + 6x = y \\ &\phi_1'(y+x) + x\phi_2'(y+x) = x \end{aligned}\right\} \text{Put } x=y.$$

$$\Rightarrow \phi_1'(2x) + x\phi_2'(2x) + \phi_2(2x) + 5x = 0 \quad -(1)$$

$$\phi_1'(2x) + x\phi_2'(2x) = x \quad -(2).$$

using (2) in (1) $\Rightarrow \phi_2(2x) = -6x = -3(2x)$

$\Rightarrow \boxed{\phi_2(x) = -3x} \Rightarrow \phi_2'(x) = -3.$

Putting in (2) $\Rightarrow \phi_1'(2x) - 3x = x$

$\phi_1'(2x) = 4x = 2(2x) \Rightarrow \phi_1'(x) = 2x$

$\underline{\phi_1(x) = x^2 + c.}$

$$\therefore \left.\begin{aligned} z &= (y+x)^2 + c - 3x(y+x) + 3x^2 \\ z &= xy. \end{aligned}\right\} \text{equating. \& put } x=y$$

we get $\boxed{c=0}$

$\therefore \boxed{z = x^2 + y^2 - xy}$

5. (c) The current i in an electric circuit is given by $i = 10e^{-t}\sin 2\pi t$ where t is in seconds. Using Newton's method, find the value of t correct to 3 decimal places for $i = 2$ amp. (10)

$\cancel{t_{n+1} = t_n - \frac{i(t_n)}{i'(t_n)}}$. Putting $i = 2$.

$$\boxed{10e^{-t}\sin 2\pi t - 2 = 0}$$

let $f(t) = 10e^{-t}\sin 2\pi t - 2$.

$$\boxed{t_{n+1} = t_n - \frac{f(t_n)}{f'(t_n)}}$$

$$f'(t) = 10\left[-e^{-t}\sin 2\pi t + 2\pi e^{-t}\cos 2\pi t\right]$$

$$= 10e^{-t}\left[2\pi\cos 2\pi t - \sin 2\pi t\right]$$

starting with $t_0 = 0$.

$t_1 = 0.03183$

$t_2 = 0.03314$

$t_3 = 0.03314$

$t_4 = 0.03314$.

$\therefore$ $\boxed{t = 0.03314 \text{ seconds.}}$

$\overline{\bar{a}+\bar{b}+c} + d + \overline{\bar{a}+e} + \bar{f}$

$ab\bar{c} + d + a\bar{e} + \bar{f}$

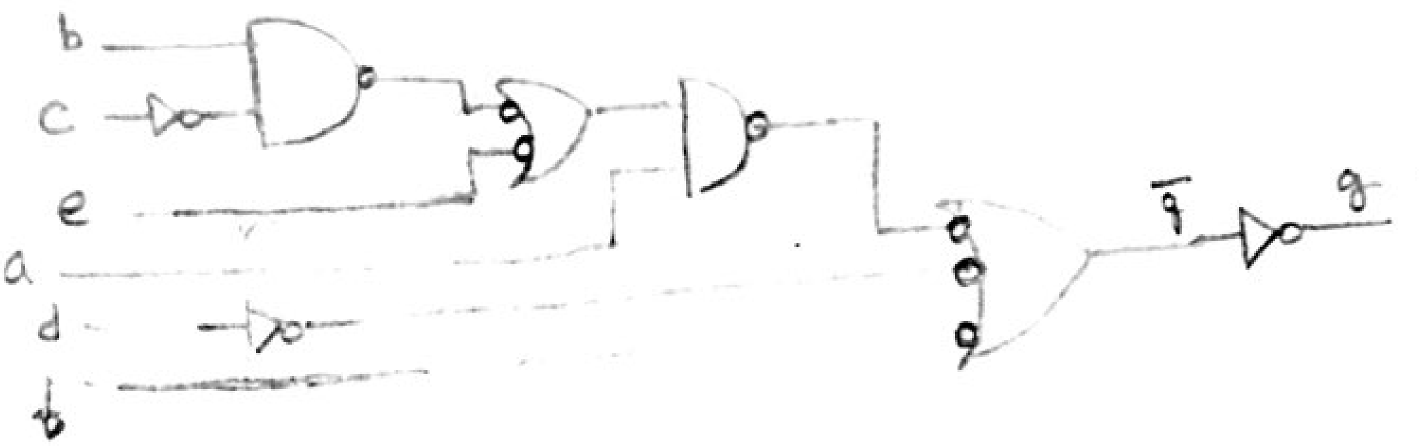
5. (d) (i) Realize the following expression by using NAND gates only.

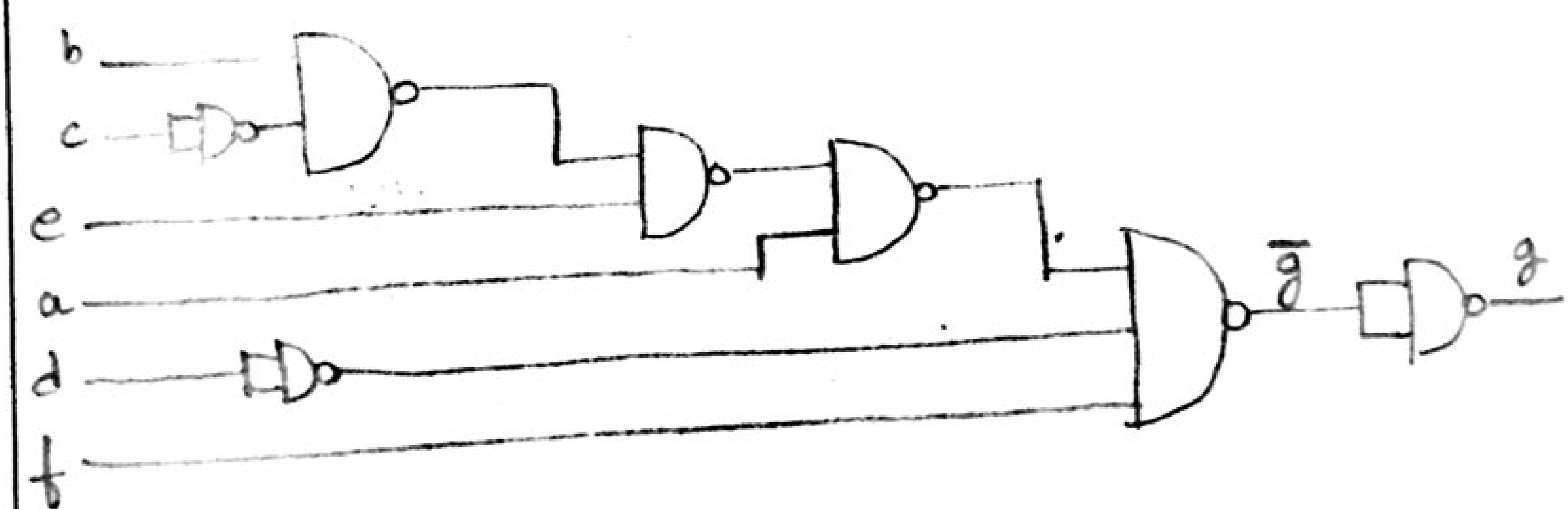
$$g = (\bar{a} + \bar{b} + c)\bar{d}\,(\bar{a} + e)f$$

where $\bar{x}$ denotes the complement of x.

(ii) Find the decimal equivalent of $(357.32)_8$ **(10)**

$\bar{g} = ab\bar{c} + d + a\bar{e} + \bar{f} = a(b\bar{c} + \bar{e}) + d + \bar{f}$



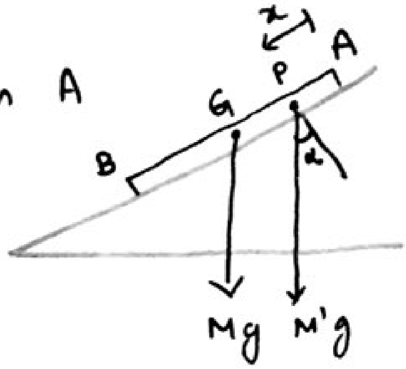


ii)

5. (e) A plank of mass M is initially at rest along a line of greatest slope of a smooth plane inclined at an angle α to the horizon, and a man of mass M', starting from the upper end, walks down the plank so that it does not move, show that he gets to the other end in time

$\sqrt{\left\{\frac{2M'a}{(M+M')g\sin\alpha}\right\}}$, where a is the length of the plane. (10)

Let $\bar{x}$ be distance of centre of gravity from A



$\Rightarrow (M+M')\bar{x} = M\frac{a}{2} + M'x$

$\Rightarrow \bar{x} = \frac{M'x + Ma/2}{M+M'}$

$\Rightarrow \ddot{\bar{x}} = \frac{M'\ddot{x}}{M+M'}$ $\Rightarrow x$

P.T.O.

6. (a) Solve $(D + D' - 1)(D + D' - 3)(D + D')z = e^{x+y} \sin(2x + y)$ **(12)**

complementary function

$$C.F = e^{x}\phi_1(y-x) + e^{3x}\phi_2(y-x) + \phi_3(y-x)$$

Particular integral

$$PI = \frac{1}{f(D,D')} e^{x+y} \sin(2x+y) = e^{x+y} \frac{1}{f(D+1, D'+1)} \sin(2x+y)$$

$$= e^{x+y} \frac{1}{(D+D'+2)} \frac{1}{D^2+2DD'+D'^2-1} \sin(2x+y) \quad D^2 \to -4, \; D'^2 \to -1, \; DD' \to -2$$

$$= \frac{e^{x+y}}{(D+D'+2)} \frac{1}{-4-4-1-1} \sin(2x+y).$$

$$= -\frac{1}{10} e^{x+y} \frac{1}{D+D'+2} \sin(2x+y)$$

$$= -\frac{1}{10} e^{x+y} \frac{(D+D'-2)\sin(2x+y)}{(D+D')^2-4}.$$

$$= -\frac{1}{10} e^{x+y} \frac{(D+D'-2)}{D^2+2DD'+(D')^2-4} \sin(2x+y)$$

$$= \frac{1}{130} e^{x+y}\left[2\cos(2x+y) + \cos(2x+y) - 2\sin(2x+y)\right]$$

$$\therefore\ z = e^{x}\phi_1(y-x) + e^{3x}\phi_2(y-x) + \phi_3(y-x)$$

$$+ \frac{1}{130} e^{x+y}\left[3\cos(2x+y) - 2\sin(2x+y)\right]$$

6. (b) Reduce $x^2r+2xys+y^2t=0$ to canonical form and hence solve it. **(13)**

$$x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} = 0$$

comparing with $a u_{xx} + b u_{xy} + c u_{yy} = 0$

$$a = x^2;\ b = 2xy;\ c = y^2$$

characteristic eqn.

$$x^2\lambda^2 + 2xy\lambda + y^2 = 0$$

$$\Rightarrow (x\lambda + y)^2 = 0 \Rightarrow x\lambda + y = 0$$

$$\Rightarrow \lambda = -y/x$$

$$\therefore \frac{dy}{dx} + \lambda = 0 \Rightarrow \frac{dy}{dx} - \frac{y}{x} = 0 \Rightarrow \frac{dy}{y} - \frac{dx}{x} = 0$$

$$\Rightarrow \log\left(\frac{y}{x}\right) = \log c_1.$$

$$B = 2a\phi_x\psi_x + b(\phi_x\psi_y + \psi_x\phi_y) + 2c\phi_y\psi_y.$$

$\Rightarrow \frac{y}{x} = c_1$

let $\xi = \phi(x, y) = y/x$; $\eta = \psi(x, y) = x$.

$\phi_x = -y/x^2$; $\phi_y = \frac{1}{x}$; $\phi_{xx} = \frac{2y}{x^3}$; $\phi_{yy} = 0$; $\phi_{xy} = -\frac{1}{x^2}$

$\psi_x = 1$; $\psi_y = 0$; $\psi_{xx} = 0 = \psi_{xy} = \psi_{yy}$.

$$A = x^2\left(\frac{y^2}{x^4}\right) + 2xy\left(\frac{-y}{x^3}\right) + y^2\left(\frac{1}{x^2}\right) = \frac{2y^2}{x^2} - \frac{2y^2}{x^2} = 0$$

$$C = x^2(1) + 2xy(0) + y^2(0) = x^2.$$

$$B = 2x^2\left(\frac{-y}{x^2}\right) + 2xy\left(0 + \frac{1}{x}\right) + 2y^2(0) = 0$$

$$R = \left(x^2\left(\frac{2y}{x^3}\right) + 2xy\left(-\frac{1}{x^2}\right) + y^2(0)\right)u_\xi \cancel{= \frac{2y}{x} - \frac{2y}{x} = 0}$$

$$+ \left(x^2(0) + 0 + 0\right)u_\eta = 0$$

The cannonical form $\Rightarrow Au_{\xi\xi} + Bu_{\xi\eta} + Cu_{\eta\eta} + R = 0$

$\therefore$ $x^2 u_{\eta\eta} = 0$. $\Rightarrow \boxed{u_{\eta\eta} = 0}$ is the cannonical form.

$u_\eta = f(\xi)$

$u = \eta f(\xi) + g(\xi)$.

$\Rightarrow \boxed{u = x f\left(\frac{y}{x}\right) + g\left(\frac{y}{x}\right)}$ is the solution.

(c) The following table gives the velocity v of a particle at time t : **(10)**

n	0	1	2	3	4	5	6
t (seconds) :	0	2	4	6	8	10	12
v (m/sec) :	4	6	16	34	60	94	136
	v_0	v_1	v_2	v_3	v_4	v_5	v_6

$S = \int v\,dt$ $\quad n = 6$

$h = 2$.

Applying simpsons 1/3rd rule

$$S = \frac{h}{3}\left[(v_0 + v_6) + 4(v_1 + v_3 + v_5) + 2(v_2 + v_4)\right]$$

$$= \frac{2}{3}\left[(4 + 136) + 4(6 + 34 + 94) + 2(16 + 60)\right]$$

$$= \underline{552\ m}.$$

6. (d) Solve the equations $27x + 6y - z = 85$; $x + y + 54z = 110$; $6x + 15y + 2z = 72$ by Gauss-Seidal method. **(15)**

Writing in matrix form $AX = B$.

$$A = \begin{bmatrix} 27 & 6 & -1 \\ 6 & 15 & 2 \\ 1 & 1 & 54 \end{bmatrix} ; \quad B = \begin{bmatrix} 85 \\ \cancel{110}\ 72 \\ \cancel{72}\ 110 \end{bmatrix}$$

Gauss-Seidal in Matrix form

$$\boxed{x^{k+1} = -(D+L)^{-1} U x^{k} + (D+L)^{-1} B}$$

$$(D+L) = \begin{bmatrix} 27 & 0 & 0 \\ 6 & 15 & 0 \\ 1 & 1 & 54 \end{bmatrix} ; \quad U = \begin{bmatrix} 0 & 6 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(D+L)^{-1} = \begin{bmatrix} 0.037 & 0 & 0 \\ -0.014 & 0.0666 & 0 \\ -4.115 \times 10^{-4} & -1.234 \times 10^{-3} & 0.01851 \end{bmatrix}$$

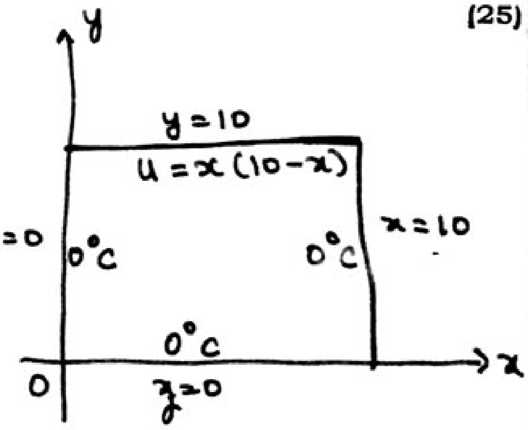
Let $x_0 = [0, 0, 0]$.

$$x_1 = \begin{bmatrix} 3.148 \\ 3.5407 \\ 1.9131 \end{bmatrix} ; \quad x_2 = \begin{bmatrix} 2.4321 \\ 3.572 \\ 1.9258 \end{bmatrix}$$

$$x_3 = \begin{bmatrix} 2.4256 \\ 3.5729 \\ 1.9259 \end{bmatrix} ; \quad x_4 = \begin{bmatrix} 2.4255 \\ 3.573 \\ 1.9259 \end{bmatrix}$$

$$\therefore \begin{cases} x = 2.4254 \\ y = 3.573 \\ z = 1.926 . \end{cases}$$

7. (a) A square plate is bounded by the lines $x = 0$, $y = 0$, $x = 10$ and $y = 10$. Its faces are insulated. The temperature along the upper horizontal edge is given by $u(x, 10) = x(10-x)$ while the other three faces are kept at 0° C. Find the steady state temperature in the plate. **(25)**



$u(x, 0) = 0$ — ④

$u(0, y) = 0$ — ⑤

$u(10, y) = 0$ — ⑥

$u(x, 10) = x(10-x)$ — ⑦.

The possible solutions are.

$u = (Ae^{px} + Be^{-px})(C\cos py + D\sin py)$ — ①

$u = (A\cos px + B\sin px)(Ce^{py} + De^{-py})$ — ②

$u = (Ax+B)(Cy+D)$ — ③.

clearly ① & ③ does not satisfy the boundary conditions

$\therefore$ The possible solution.

$u(x,y) = (A\cos px + B\sin px)(Ce^{py} + De^{-py})$

using ⑤ $\Rightarrow$ $\underline{0 = A}$.

using ⑥ $\Rightarrow B\sin 10p = 0 \Rightarrow \sin 10p = 0$

$\Rightarrow 10p = n\pi \Rightarrow \boxed{p = \frac{n\pi}{10}}$; $n = 1, 2, 3, \dots$

using ④ $\Rightarrow C + D = 0 \Rightarrow \underline{D = -C}$.

$\therefore u(x,y) = 2BC \sin px \cdot \sinh(py)$

$$\boxed{u(x,y) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{10}x\right)\sinh\left(\frac{n\pi}{10}y\right)}$$

using ⑦

$$x(10-x) = \sum_{n=1}^{\infty}\{b_n \sinh(n\pi)\} \sin\left(\frac{n\pi}{10}x\right).$$

$$\Rightarrow 10x - x^2 = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{10}x\right).$$

This is sine half-range fourier series in $(0, 10)$

$$B_n = \frac{2}{10}\int_0^{10}(10x - x^2)\sin\left(\frac{n\pi}{10}x\right)dx$$

$$= \frac{2}{10}\left[(10x-x^2)\left(-\frac{\cos kx}{k}\right) - (10-2x)\left(-\frac{\sin kx}{k^2}\right) + (-2)\left(+\frac{\cos kx}{k^3}\right)\right]_0^{10} \quad \text{where } k = \frac{n\pi}{10}$$

$$= \frac{2}{10}\left[(-10)\frac{\sin 10k}{k^2} - 2\frac{\cos 10k}{k^3} - \left(-\frac{2}{k^3}\right)\right]$$

$$= \frac{2}{10}\left[\frac{-2\cos n\pi + 2}{k^3}\right] \quad \text{where } k = \frac{n\pi}{10}.$$

$$= \frac{4}{10k^3}(1 - \cos n\pi) = \begin{cases} 0; & n \text{ is even.} \\ \frac{8}{10k^3}; & n \text{ is odd} \end{cases}.$$

$$\Rightarrow b_n = \frac{8}{10k^3\sinh(n\pi)} \quad \text{where } n \text{ is odd.}$$

$$\therefore u(x,y) = \frac{800}{\pi^3}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^3}\frac{1}{\sinh[(2m-1)\pi]}$$

$$\times \sin\left(\frac{(2m-1)\pi}{10}x\right)\sinh\left(\frac{(2m-1)\pi}{10}y\right)$$

is the required solution.

7. (b) Using modified Euler's method, find an approximate value of y when $x = 0.3$, given that $dy/dx = x + y$ and $y = 1$ when $x = 0$.

(10)

$f(x,y) = x + y$; $x_0 = 0$; $y_0 = 1$; $h = 0.3$; $x_1 = 0.3$

$y_1 = y(x_1) = y(0.3) = ?$

$y_1^{(0)} = y_0 + h f(x_0, y_0)$

$= 1 + 0.3(0+1) = 1.3$.

$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$

$= 1 + \frac{0.3}{2}\left[1 + 0.3 + y_1^{(0)}\right] = 1.39$

$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right] = 1.4035$

$y_1^{(3)} = 1.4055$

$y_1^{(4)} = 1.4058$

$y_1^{(5)} = 1.4058$.

$\therefore\ y_1 = y(x_1) = y(0.3) = 1.4058$.

7. (c) For the given set of data points

$(x_1, f(x_1)), (x_2, f(x_2)), \ldots\ldots\ldots(x_n, f(x_n))$

write an algorithm to find the value of $f(x)$ by using Lagrange's interpolation formula. **(15)**

Step 1 : Enter xi, f(xi) for i = 1 to n.

step 2 : Sum = 0.

step 3 : for i = 1 to n in steps of 1.

step 4 : prod = 1

step 5 : for j = 1 to n in steps of 1

step 6 : if i = j
then j = j + 1

Step 7: prod = prod × $\frac{x - x_j}{x_i - x_j}$

step 8 : end for.

step 9 : sum = sum + prod · $f(x_i)$

step 10 : end for.

step 11 : print "sum"

8. (a) Determine the motion, of a spherical pendulum, by using Hamilton's equations. **(16)**