A CONSOLIDATED QUESTION PAPER-CUM-ANSWER BOOKLET

IMS™

(INSTITUTE OF MATHEMATICAL SCIENCES)

TEST SERIES (MAIN)-2014

Test Code: FULL TEST P-I(M) IAS / T- 09

MATHEMATICS

by K. VENKANNA

The person with 14 years of Teaching Experience

FULL TEST P-I

Time: Three Hours — *Maximum Marks: 250*

INSTRUCTIONS

1. This question paper-cum-answer booklet has 52 pages and has 34 PART/SUBPART questions. Please ensure that the copy of the question paper-cum-answer booklet you have received contains all the questions.
2. Write your Name, Roll Number, Name of the Test Centre and Medium in the appropriate space provided on the right side.
3. A consolidated Question Paper-cum-Answer Booklet, having space below each part/sub part of a question shall be provided to them for writing the answers. Candidates shall be required to attempt answer to the part/sub-part of a question strictly within the pre-defined space. Any attempt outside the pre-defined space shall not be evaluated.
4. Answer must be written in the medium specified in the admission Certificate issued to you, which must be stated clearly on the right side. No marks will be given for the answers written in a medium other than that specified in the Admission Certificate.
5. Candidates should attempt Question Nos. 1 and 5, which are compulsory, and any THREE of the remaining questions selecting at least ONE question from each Section.
6. The number of marks carried by each question is indicated at the end of the question. Assume suitable data if considered necessary and indicate the same clearly.
7. Symbols/notations carry their usual meanings, unless otherwise indicated.
8. All questions carry equal marks.
9. All answers must be written in blue/black ink only. Sketch pen, pencil or ink of any other colour should not be used.
10. All rough work should be done in the space provided and scored out finally.
11. The candidate should respect the instructions given by the invigilator.
12. The question paper-cum-answer booklet must be returned in its entirety to the invigilator before leaving the examination hall. Do not remove any page from this booklet.

READ INSTRUCTIONS ON THE LEFT SIDE OF THIS PAGE CAREFULLY

Name: Nitish.K

Roll No.: 149709

Test Centre: Bangalore

Medium: English

Do not write your Roll Number or Name anywhere else in this Question Paper-cum-Answer Booklet.

I have read all the instructions and shall abide by them

Nitish.K

Signature of the Candidate

I have verified the information filled by the candidate above

Signature of the invigilator

IMPORTANT NOTE:
Whenever a question is being attempted, all its parts/ sub-parts must be attempted contiguously. This means that before moving on to the next question to be attempted, candidates must finish attempting all parts/ sub-parts of the previous question attempted. This is to be strictly followed.

INDEX TABLE

QUESTION	No.	PAGE NO.	MAX. MARKS	MARKS OBTAINED
1	(a)			
	(b)			
	(c)			
	(d)			
	(e)			
2	(a)			
	(b)			
	(c)			
	(d)			
3	(a)			
	(b)			
	(c)			
	(d)			
4	(a)			
	(b)			
	(c)			
	(d)			
5	(a)			
	(b)			
	(c)			
	(d)			
	(e)			
6	(a)			
	(b)			
	(c)			
	(d)			
7	(a)			
	(b)			
	(c)			
	(d)			
8	(a)			
	(b)			
	(c)			
	(d)			
			Total Marks	

SECTION-A

1. (a) If we shift to A-7*I*, what are the eigenvalues and eigenvectors and how are they related to those of A?

$$B = A - 7I = \begin{bmatrix} -6 & -1 \\ 2 & -3 \end{bmatrix}$$

(10)

if λ is an eigenvalue of A $\Rightarrow AX = \lambda X$

$BX = (A-7I)X = AX - 7X = \lambda X - 7X = (\lambda - 7)X$

$\Rightarrow$ if λ is an eigenvalue of A $\Leftrightarrow$ $(\lambda - 7)$ is an eigenvalue of $A - 7I$.

$\Rightarrow$ eigenvectors of Both A and $A-7I$ are same.

$$A = B + 7I = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & -1 \\ 2 & 4-\lambda \end{vmatrix} = 0 \Rightarrow \underline{\lambda = 3, 2.}$$

$\therefore$ eigen values of A are 3 & 2

$\Rightarrow$ eigen values of $A - 7I$ are -4 & -5.

$x_1 = [-1, 2]^T$ & $x_2 = [-1, 1]^T$ are eigenvectors ~~are eigenvectors~~ of A corresponding to $\lambda = 3$ & $\lambda = 2$.

$\Rightarrow$ $x_1 = [-1, 2]^T$ & $x_2 = [-1, 1]^T$ are also eigenvectors of $B = A - 7I$ corresponding to eigenvalues $\lambda = -4$ & $\lambda = -5$ respectively.

(b) Let V be the set of all complex-valued functions f on the real line such that (for all t in R)

$f(-t) = \overline{f(t)}$,

the bar denotes complex conjugation. Show that V, with operations $(f+g)(t) = f(t)+g(t)$

$(cf)(t) = cf(t)$

is a vector space over the field of real numbers. (10)

① ~~$(f+g)(-t) = (\overline{f+g})(-t) = (\bar{f}+\bar{g})(-t)$~~

① $(f+g)(-t) = f(-t) + g(-t) = \bar{f}(t) + \bar{g}(t)$

$(g+f)(-t) = g(-t) + f(-t) = \bar{g}(t) + \bar{f}(t)$

$\Rightarrow f+g = g+f$.

② clearly $0 + f = f = f + 0$

$\Rightarrow$ 0 is the additive identity.

③ let ~~I(x)=x~~ $I(t) = t$

$(f \cdot I)(-t) = f(I(-t)) = f(-t)$

$\Rightarrow f \cdot I = f = I \cdot f$ $\Rightarrow$ $I(t) = t$ is the multiplicative identity.

④ composition of functions is anyway associative.

⑤ ~~$(a+b)f(-t) = (a+b)$~~

$((a+b)f)(-t) = (a+b)f(-t) = (a+b)\bar{f}(t)$.

$af(-t) + bf(-t) = a\bar{f}(t) + b\bar{f}(t) = (a+b)\bar{f}(t)$.

$\Rightarrow (a+b)f = af + bf.$

(7) |||ly $a(f+g) = af + ag.$

~~(8) $fg(-t) = f(g(-t)) = f(g(t)) = \bar{f}(\bar{g}(t))$~~

~~$gf(-t) = g($~~

Hence V is a vector space.

(c) Determine the values of A and B for which

$\underset{x \to 0}{Lt} \frac{\sin 3x + A \sin 2x + B \sin x}{x^3}$ exists and find the limit (10)

$$I = \lim_{x \to 0} \frac{3\cos 3x + 2A\cos 2x + B\cos x}{5x^4}$$

$$\Rightarrow \boxed{3 + 2A + B = 0} \quad \text{— (1)}$$

$$= \lim_{x \to 0} \frac{-9\sin 3x - 4A\sin 2x - B\sin x}{20x^3} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{-27\cos 3x - 8A\cos 2x - B\cos x}{60x^2}$$

$$\Rightarrow \boxed{27 + 8A + B = 0} \quad - (2)$$

$$= \lim_{x \to 0} \frac{81 \sin 3x + 16A \sin 2x + B \sin x}{120x}.$$

$$= \lim_{x \to 0} \frac{243 \cos 3x + 32A \cos 2x + B \cos x}{120}$$

$$= \frac{243 + 32A + B}{120}$$

Solving (1) & (2) $\Rightarrow$ $A = -4$; $B = 5$

$$\therefore \quad I = \frac{243 + 32(-4) + 5}{120} = 1 \,//$$

$$\therefore \lim_{x \to 0} \frac{\sin 3x + A \sin 2x + B \sin x}{x^5} = 1 \,//.$$

& $A = -4$

$B = 5$.

(d) **Investigate the maxima and minima of**

$f(x, y) = x^2 + 3xy + y^2 + x^3 + y^3$ **(10)**

$f(x,y) = x^2 + 3xy + y^2 + x^3 + y^3$

$f_x = 2x + 3y + 3x^2$; $f_y = 3x + 2y + 3y^2$

$f_{xx} = 2 + 6x$; $f_{xy} = 3$; $f_{yy} = 2 + 6y$.

For stationary points $\Rightarrow f_x = 0$ & $f_y = 0$

$$\Rightarrow \begin{cases} 3x^2 + 2x + 3y = 0 & -① \\ 3y^2 + 3x + 2y = 0 & -② \end{cases}$$

subtracting $\Rightarrow 3(x^2 - y^2) - (x - y) = 0$

$\Rightarrow (x-y)\left[3(x+y) - 1\right] = 0$.

$\Rightarrow x - y = 0$ or $x + y = 1/3$.

I) $x = y \Rightarrow 3x^2 + 5x = 0 \Rightarrow x = 0$ or $x = -5/3$

$(0,0)$ & $(-5/3, -5/3)$.

II) $y = \frac{1}{3} - x \Rightarrow 3x^2 - x + 1 = 0$.
This has no real solutions.

At $(0,0) \Rightarrow f_{xx} = 2$; $f_{xy} = 3$; $f_{yy} = 2 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -5 < 0$

$\Rightarrow f_{xx}f_{yy} - f_{xy}^2 < 0 \Rightarrow$ $\boxed{\text{No maxima nor minima}}$.

At $(-5/3, -5/3) \Rightarrow f_{xx} = -8$; $f_{xy} = 3$; $f_{yy} = -8$

$f_{xx}f_{yy} - f_{xy}^2 = 64 - 3^2 = 55 > 0$ & $f_{xx} < 0$.

$\therefore f$ has $\boxed{\text{maxima}}$ at $(-5/3, -5/3)$.

(e) Find the angle between the lines given by $x+y+z=0$ and

$\frac{yz}{q-r}+\frac{zx}{r-p}+\frac{xy}{p-q}=0.$ let $a=\frac{1}{q-r}$; $b=\frac{1}{r-p}$; $c=\frac{1}{p-q}$ (10)

$\Rightarrow ayz + bzx + cxy = 0$ — ①

$x+y+z=0$ — ②.

Let the eqn of any line of section be

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$$

$\Rightarrow amn + bnl + clm = 0$ & $l+m+n=0$

$\Rightarrow -am(l+m) - bl(m+l) + clm = 0$

$\Rightarrow b\left(\frac{l}{m}\right)^2 + (a+b-c)\frac{l}{m} + a = 0.$

$\frac{l_1}{m_1}\cdot\frac{l_2}{m_2} = \frac{a}{b} \Rightarrow \frac{l_1 l_2}{a} = \frac{m_1 m_2}{b} = \frac{n_1 n_2}{c} = \lambda$ (say)

$\frac{l_1}{m_1}+\frac{l_2}{m_2} = \frac{c-a-b}{b} \Rightarrow \frac{l_1 m_2 + l_2 m_1}{c-a-b} = \frac{m_1 m_2}{b} = \lambda.$

$$(l_1 m_2 - l_2 m_1)^2 = (l_1 m_2 + l_2 m_1)^2 - 4 l_1 l_2 m_1 m_2$$

$$= \lambda^2 (c-a-b)^2 - 4ab\lambda^2$$

$$= \lambda^2\left[a^2+b^2+c^2-2ab-2bc-2ca\right]$$

$\therefore \tan^2\theta = \frac{\sum (l_1 m_2 - l_2 m_1)^2}{(l_1 l_2 + m_1 m_2 + n_1 n_2)^2} = \frac{3\lambda^2(a^2+b^2+c^2-2ab-2bc-2ca)}{\lambda^2(a+b+c)^2}$

$\frac{\tan^2\theta}{3} - 1 = \frac{-4ab-4bc-4ca}{(a+b+c)^2} = \frac{-4abc\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}{(a+b+c)^2} = 0$

$\Rightarrow \tan^2\theta = 3 \Rightarrow \tan\theta = \sqrt{3} \Rightarrow \boxed{\theta = \pi/3}$ $\because \boxed{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0}$

2. (a) (i) Let $T : P_3[0, 1] \rightarrow P_2[0, 1]$ be defined by $(T_p)(X) = P''(X) + P'(X)$. Then find the matrix representation of T with respect to the basis $\{1, x, x^2, x^3\}$ and $\{1, x, x^2\}$ of $P_3[0, 1]$ and $P_2[0, 1]$ respectively.

(ii) If $A = \begin{bmatrix} 1 & 0 & 0 \\ i & \frac{-1+i\sqrt{3}}{2} & 0 \\ 0 & 1+2i & \frac{-1-\sqrt{3}i}{2} \end{bmatrix}$ then find trace of A^{102}.

(20)

(i) $T(p(x)) = p''(x) + p'(x)$.

$T(1) = 0 = 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2$

$T(x) = 0 + 1 = 1 \cdot 1 + 0 \cdot x + 0 \cdot x^2$

$T(x^2) = 2x + 2 = 2 \cdot 1 + 2 \cdot x + 0 \cdot x^2$

$T(x^3) = 3x^2 + 6x = 0 \cdot 1 + 6 \cdot x + 3 \cdot x^2$.

$$[T] = \begin{bmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 3 \end{bmatrix}_{3\times4}.$$

(ii) $A = \begin{bmatrix} 1 & 0 & 0 \\ i & \omega & 0 \\ 0 & 1+2i & \omega^2 \end{bmatrix}$ where $1, \omega, \omega^2$ are cube roots of units.

$$A = \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix}$$

Then $\underline{\text{trace } A^n = a^n + c^n + f^n}$.

$\therefore$ trace $A^{102} = a^{102} + c^{102} + f^{102}$

where $a = 1$, $c = \omega$; $f = \omega^2$

trace $A^{102} = 1 + \omega^{102} + \omega^{204}$

But $\omega^3 = 1$

$\Rightarrow$ trace $A^{102} = 1 + \omega^{34} + \omega^{68}$

$= 1 + \omega \cdot \omega^{33} + \omega^2 \cdot \omega^{66}$

$= 1 + \omega \cdot (\omega^3)^{11} + \omega^2 \cdot (\omega^3)^{22}$

$= 1 + \omega + \omega^2$

$= \underline{\underline{0}}$

as sum of cubic roots of unity is zero.

2 (b) Show that the function of $f : \mathbb{R}^2 \to R$ defined by setting

$$f(x,y) = \begin{cases} x\sin(1/x) + y\sin(1/y), & \text{when } xy \neq 0 \\ x\sin(1/x) & \text{, when } x \neq 0\ y = 0 \\ y\sin(1/y) & \text{, when } x = 0\ y \neq 0 \\ 0 & \text{, when } x = y = 0 \end{cases}$$

is continuous but not differentiable at (0, 0). (15)

$$|f(x,y) - f(0,0)| = \left|x \sin\left(\frac{1}{x}\right) + y \sin\left(\frac{1}{y}\right)\right|$$

$$\leq |x| + |y| < \epsilon$$

if $|x| < \frac{\epsilon}{2} = \delta$ & $|y| < \frac{\epsilon}{2} = \delta$.

$\therefore |f(x,y) - f(0,0)| < \epsilon$ when $|x-0| < \delta$ & $|y-0| < \delta$

$\therefore f$ is continuous at $(0,0)$.

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h}$$

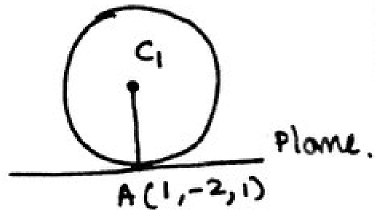
$$= \lim_{h\to 0} \sin\left(\frac{1}{h}\right) \Rightarrow \text{does not exists.}$$

111'y $f_y(0,0)$ does not exist.

As partial derivatives f_x and f_y does not exist at $(0,0) \Rightarrow f$ cannot be differentiable at $(0,0)$

(c) Find the equation of the sphere which touches the plane $3x+2y-z+2=0$ at the point $(1, -2, 1)$ and cuts orthogonally the sphere $x^2+y^2+z^2-4x+6y+4=0$. **[15]**

drs of $AC_1 \Rightarrow 3, 2, -1$.



$\therefore$ eqn of AC_1

$$\frac{x-1}{3} = \frac{y+2}{2} = \frac{z-1}{-1} = r.$$

Any point of this line is

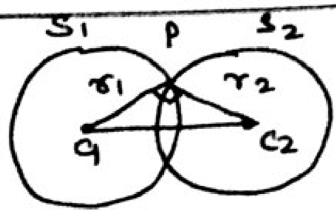
$(1+3r, -2+2r, 1-r)$

let this be C_1.

radius $r_1 = \sqrt{(3r)^2+(2r)^2+(-r)^2} = \sqrt{14r^2}$

if S_1 & S_2 cut orthogonally then

$$\underline{r_1^2 + r_2^2 = C_1C_2^2}.$$



$$C_2 = (2, -3, 0).$$

$$r_2 = \sqrt{4+9-4} = 3.$$

$$\therefore 14r^2 + 9 = (3r-1)^2 + (2r+1)^2 + (-r+1)^2$$

$$9 = -6r + 4r - 2r + 1 + 1 + 1$$

$$4r = 3 - 9 = -6 \Rightarrow \boxed{r = -3/2}$$

$$\therefore r_1^2 = \frac{63}{2} \quad \& \quad C_1 = \left(-\frac{7}{2}, -5, \frac{5}{2}\right).$$

$\therefore$ equation of the required sphere.

$$\boxed{\left(x+\frac{7}{2}\right)^2 + (y+5)^2 + \left(z-\frac{5}{2}\right)^2 = \frac{63}{2}}$$

4. (a) (i) Show that $A^2 = 0$ is possible but $A^TA = 0$ is not possible.

(ii) Verify that $(AB)^T$ equals $B^T A^T$ but those are different from $A^T B^T$:

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \quad AB = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$$

In case AB=BA, how do you prove that $B^T A^T = A^T B^T$? (15)

(i) Let $A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \Rightarrow A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

But ~~let~~ $A^T = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$

$\Rightarrow A^T A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \neq 0.$

(ii) $A^T = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$; $B^T = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$;

$$(AB)^T = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$$

$$B^T A^T = \cancel{\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}} \cancel{\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$$

$$A^T B^T = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 2 \\ 3 & 1 \end{bmatrix} \neq (AB)^T$$

Let $AB = BA$

Taking transposes both sides.

$$(AB)^T = (BA)^T$$

$$\Rightarrow \underline{B^T A^T = A^T B^T}$$

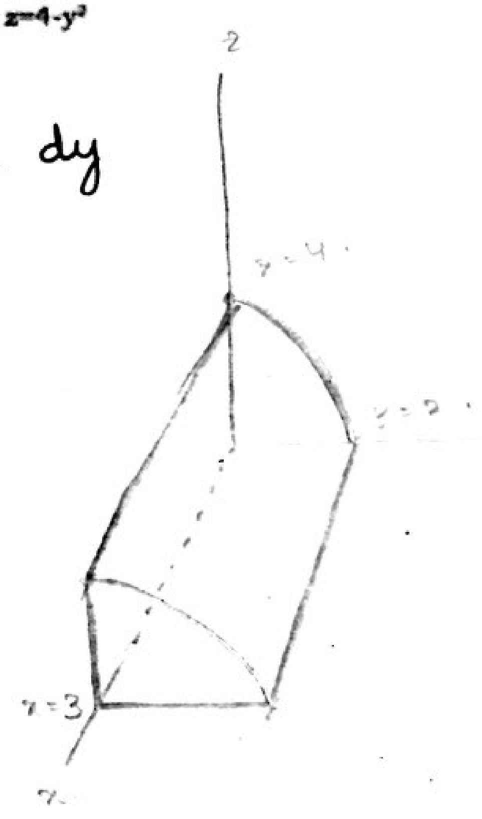
4 (b) Find the volume of the solid in the first octant bounded by the coordinate planes, the plane x=3 and the parabolic cylinder z=4-y² (15)

$$V = \int_{x=0}^{3} \int_{y=0}^{2} \int_{z=0}^{4-y^2} dz \cdot dx\ dy$$

$$= \int_{x=0}^{3} \int_{y=0}^{2} (4-y^2)\, dy\, dx$$

$$= \int_{x=0}^{3} \frac{16}{3}\, dx$$

$$= 16$$



$\therefore$ $\boxed{\text{Volume} = 16 \text{ units}}$

4 (c) CP, CQ are any two conjugate semi-diametrs of the ellipse $(x^2/a^2) + (y^2/b^2) = 1$, $z=c$, CP', CQ' are the conjugate diameters of the allipse $(x^2/a^2) + (y^2/b^2) = 1, z = -c$ drawn in the same directions as CP and CQ. Prove that the hyperboloid $(2x^2/a^2) + (2y^2/b^2) - (z^2/c^2) = 1$ is generated by either PQ' or $P'Q'$.

(20)

$P = (a\cos\alpha, b\sin\alpha, c)$; $Q = (-a\sin\alpha, b\cos\alpha, c)$

$p' = (a\cos\alpha, b\sin\alpha, -c)$; $Q' = (-a\sin\alpha, b\cos\alpha, -c)$

equation of PQ'

$$\frac{x - a\cos\alpha}{-a\sin\alpha - a\cos\alpha} = \frac{y - b\sin\alpha}{b\cos\alpha - b\sin\alpha} = \frac{z - c}{-c-c}.$$

$$\Rightarrow \frac{x/a - \cos\alpha}{-(\sin\alpha + \cos\alpha)} = \frac{y/b - \sin\alpha}{\cos\alpha - \sin\alpha} = \frac{z/c - 1}{-2} = r \text{ (say)}$$

$\Rightarrow \frac{x}{a} = \cos\alpha - r(\sin\alpha + \cos\alpha)$ — (1) & $\frac{z}{c} = 1 - 2r$ — (3)

$\frac{y}{b} = \sin\alpha + r(\cos\alpha - \sin\alpha)$ (2)

squaring & adding (1) & (2)

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 + r^2(1+1+0) - 2r\left[+1\right] = 1 + 2r^2 + 2r.$$

Hence.

$$\Rightarrow 2\left(\frac{x}{a}\right)^2 + 2\left(\frac{y}{b}\right)^2 = 4r^2 + 4r + 2$$

$$= 4r^2 + 4r + 1 + 1 = (2r+1)^2 + 1$$

$$= \left(\frac{z}{c}\right)^2 + 1 \qquad \text{using (3).}$$

$$\Rightarrow \boxed{2\frac{x^2}{a^2} + 2\frac{y^2}{b^2} - \frac{z^2}{c^2} = 1}$$

lll'y it can be shown that P'Q also generates the same hyperboloid.

SECTION-B

5. (a) Solve $(3 + 2 \sin x + \cos x)\, dy = (1 + 2 \sin y + \cos y)\, dx$. (10)

(b) Solve $d^2y/dx^2+2(dy/dx)+10y+37\sin 3x=0$ and find the value of y when $x=\pi/2$ if it given that $y=3$ and $dy/dx=0$ when $x=0$. (10)

$$y''+2y'+10y=-37\sin 3x.$$

~~applying Laplace transforms.~~

$$\cancel{s^2L(y)-sy(0)-y'(0)+2[sL(y)-y(0)]+10L(y)=-37\cdot\frac{3}{s^2+9}.}$$

$$\cancel{(s^2+2s+10)L(y)-3s-6=\frac{111}{s^2+9}.}$$

$$(D^2+2D+10)y=-37\sin 3x. \Rightarrow m=-1\pm 3i$$

$$CF=e^{-x}(c_1\cos 3x+c_2\sin 3x)$$

$$PI=\frac{1}{D^2+2D+10}(-37\sin 3x) \qquad D^2\to -3^2$$

$$=\frac{1}{2D+1}(-37\sin 3x)=\frac{2D-1}{4D^2-1}(-37\sin 3x)$$

$$=\frac{-37}{-37}[6\cos 3x-\sin 3x]$$

$$\therefore y(x)=e^{-x}[c_1\cos 3x+c_2\sin 3x]+6\cos 3x-\sin 3x.$$

$$x=0;\ y=3 \Rightarrow 3=c_1+6 \Rightarrow \boxed{c_1=-3}$$

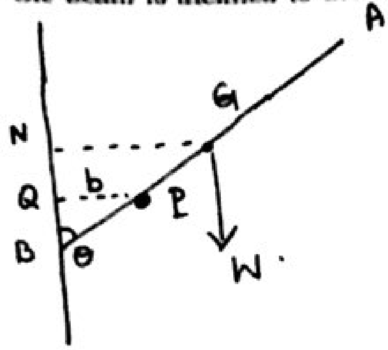
$$y'(x)=e^{-x}[-3c_1\sin 3x+3c_2\cos 3x]-e^{-x}[c_1\cos 3x+c_2\sin 3x]$$

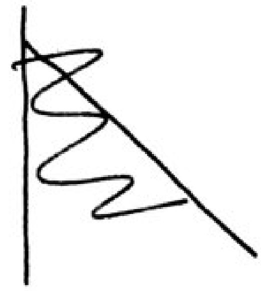
$$-18\sin 3x-3\cos 3x$$

$$y'=0 \text{ when } x=0 \Rightarrow \boxed{c_2=0} \Rightarrow \boxed{y(x)=e^{-x}(-3\cos 3x)+6\cos 3x-\sin 3x}$$

$$\therefore y(\pi/2)=-\sin(3\pi/2)=-(-1)=1 \qquad \therefore \boxed{y(\pi/2)=1}$$

(c) A uniform beam of length 2a, tests in equilibrium against a smooth vertical wall and upon a smooth peg at a distance b from the wall. Show that in the position of equilibrium the beam is inclined to the wall at an angle $\sin^{-1}(b/a)^{1/3}$ [10]





$NQ = NB - QB = a\cos\theta - b\cot\theta$

Give a small displacement such that θ changes θ+δθ. By principle of virtual work

$-W\delta(QN) = 0$

$\Rightarrow \delta(a\cos\theta - b\cot\theta) = 0$

$\Rightarrow -a\sin\theta + b\,\mathrm{cosec}^2\theta = 0$

$\Rightarrow a\sin\theta = b\,\mathrm{cosec}^2\theta$

$\Rightarrow \sin^3\theta = b/a$

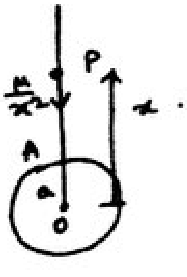
$\Rightarrow \boxed{\theta = \sin^{-1}(b/a)^{1/3}}$

P.T.O.

(d) A particle is projected vertically upwards from the surface of earth with a velocity just sufficient to carry it to the infinity. Prove that the time it takes to reach a height h is

$$\frac{1}{3}\sqrt{\left(\frac{2a}{g}\right)}\left[\left(1+\frac{h}{a}\right)^{3/2}-1\right].$$

Where a is the radius of the earth. (10)



equation of motion of the particle at P. $\Rightarrow \frac{d^2x}{dt^2} = -\frac{\mu}{x^2}$.

at A $\Rightarrow \frac{d^2x}{dt^2} = -g$ & $x = a$ $\Rightarrow -\frac{\mu}{a^2} = -g \Rightarrow \mu = a^2 g$

$\therefore \frac{d^2x}{dt^2} = -\frac{a^2 g}{x^2} \Rightarrow v^2 = \left(\frac{dx}{dt}\right)^2 = \frac{2a^2 g}{x} + B$.

But when $x \to \infty$; $v \to 0$ $\Rightarrow B = 0$.

$\therefore \frac{dx}{dt} = \sqrt{\frac{2a^2 g}{x}} \Rightarrow dt = \frac{\sqrt{x}\, dx}{\sqrt{2a^2 g}}$.

$\Rightarrow t = \frac{1}{\sqrt{2a^2 g}} \int_{x=a}^{a+h} x^{1/2}\, dx = \frac{2}{3\sqrt{2a^2 g}} \left[(a+h)^{3/2} - a^{3/2}\right]$

$t = \frac{1}{3a}\sqrt{\frac{2}{g}}\, a^{3/2} \left[\left(1 + \frac{h}{a}\right)^{3/2} - 1\right]$

$$\boxed{t = \frac{1}{3}\sqrt{\frac{2a}{g}} \cdot \left[\left(1 + \frac{h}{a}\right)^{3/2} - 1\right]}$$

Hence proved.

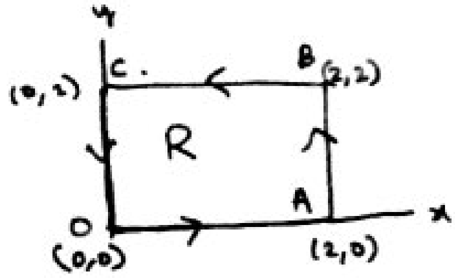
(e) Prove the Frenet-serret-formula. (i) $\frac{dT}{dS} = \kappa N$ (ii) $\frac{dB}{dS} = \tau N$ (iii) $\frac{dN}{dS} = \tau B - \kappa \tau$

(10)

P.T.O.

8. (a) Verify Green's theorem in the plane for $\int_C (x^2 - xy^3)dx + (y^2 - 2xy)dy$, where C is the square with vertices (0,0), (2,0), (2,2), (0,2). (12)

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$



① Along OA ⇒ $y = 0$

$$\int_{OA} x^2\,dx = \int_{x=0}^{2} x^2 = \frac{8}{3}$$

② Along AB ⇒ $x = 2$; $dx = 0$

$$\int_{AB} (y^2 - 4y)\,dy = -\frac{16}{3}.$$

P.T.O.

(3) Along BC $\Rightarrow$ $y=2$; $dy=0$

$$\int_{BC} (x^2 - 8x)\,dx = \frac{40}{3}$$

(4) Along CO $\Rightarrow$ $x=0$; $dx=0$

$$\int_{CO} y^2\,dy = \left.\frac{y^3}{3}\right]_{y=2}^{0} = -\frac{8}{3}.$$

$$\Rightarrow \oint_C M\,dx + N\,dy = \frac{8}{3} - \frac{16}{3} + \frac{40}{3} - \frac{8}{3} = 8 \,//.$$

$$\frac{\partial M}{\partial y} = -3xy^2; \quad \frac{\partial N}{\partial x} = -2y.$$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \int_{x=0}^{2}\int_{y=0}^{2} (-2y + 3xy^2)\,dy\,dx$$

$$= \int_{x=0}^{2} (-4 + x\,8)\,dx = -4(2) + 4(4) = 16 - 8$$

$$= 8 \,//$$

$$\therefore \oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dx.$$

(b) (i) Show that $q.\nabla q = \frac{1}{2}\nabla q^2 - q \times curl\, q$

div. curl $(curl\, \bar{a}\, \phi) + \nabla^2 div(\bar{a}\phi) = \bar{a}\,.\, grad\, \nabla^2 \phi$ where ϕ is a scalar point function. (12)

P.T.O.

(c) If $\phi = 2xyz^2, F = xy\hat{i} - z\hat{j} + x^2\hat{k}$ and C is the Curve $x = t^2, y = 2t, z = t^3$ from $t = 0$ to $t = 1$, evaluate the line integrals. (i) $\int_c \phi\, dr$ (ii) $F \times dr$ (08)

(i) $\int_C \phi \cdot dr = \int_C 2xyz^2 (dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$

$= \int_{t=0}^{1} 4t^9 (2t\,\hat{i} + 2\hat{j} + 3t^2\,\hat{k})\, dt$

$= \int_{t=0}^{1} (8t^{10}\,\hat{i} + 8t^9\,\hat{j} + 12t^{11}\,\hat{k})\, dt$.

$= \frac{8}{11}\hat{i} + \frac{8}{10}\hat{j} + \hat{k}$ //.

(ii) $\vec{F} \times dr = [\hat{i}(-3t^5 - 2t^4) - \hat{j}(4t^5) + \hat{k}(4t^3 + 2t^4)]\, dt$

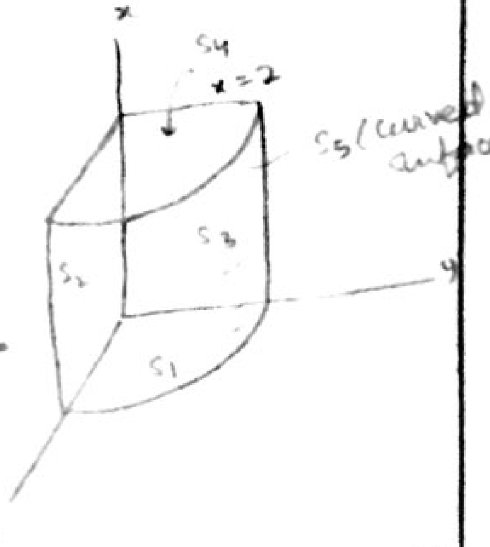
$\int_0^1 F \times dr = \int_0^1 \hat{i}(-3t^5 - 2t^4) - \hat{j}(4t^5) + \hat{k}(4t^3 + 2t^4)\, dt$

$= \hat{i}\left(-\frac{3}{6} - \frac{2}{5}\right) - \hat{j}\left(\frac{4}{6}\right) + \hat{k}\left(\frac{4}{4} + \frac{2}{5}\right)$

$= -\frac{9}{10}\hat{i} - \frac{2}{3}\hat{j} + \frac{7}{5}\hat{k}$ //.

(d) Verify the divergence theorem for A $2x^2 y \hat{i} - y^2 \hat{j} + 4xz^2 \hat{k}$ taken over the region in the first octant bounded by $y^2 + Z^2 = 9$ and $x=2$. (18)

$$\iint_S \vec{A}\cdot\hat{n}\, dS = \iiint_V \operatorname{div}\vec{A}\, dV .$$



on $S_1 \Rightarrow x = 0$ & $\hat{n} = -\hat{i}$

$$\iint_{S_1} \vec{A}\cdot\hat{n}\, ds = \iint_{S_1} -2x^2 y\, dy\, dz = \underline{0}$$

$\cancel{\text{on } S_4 \Rightarrow x=2; \quad \hat{n} = \hat{i}}$

$$\cancel{\iint_{S_4} \vec{A}\cdot\hat{n}\, ds = \iint_{S_4} 2\cdot(4) y\, dz\, dy}.$$

Put $\cancel{y = r\cos\theta}$ $\cancel{z = r\sin\theta}$

$$= \cancel{8\int_{\theta=0}^{\pi/2}\int_{r=0}^{3} r\cos\theta\, r\, dr\, d\theta} = \cancel{8\cdot\frac{3^3}{3}} = \cancel{8\times 9 = 72}$$

on $S_2 \Rightarrow y = 0$ & $\hat{n} = -\hat{j}$

$$\iint_{S_2} \vec{A}\cdot\hat{n}\, ds = \underline{0}$$

on $S_3 \Rightarrow z = 0$ & $\hat{n} = -\hat{k}$.

$$\iint_{S_3} \vec{A}\cdot\hat{n}\, ds = \underline{0}$$

$\underline{\text{on } S_4}$ $\Rightarrow x = 2; \quad \hat{n} = \hat{i} \quad \Rightarrow \iint_{S_4} \vec{A}\cdot\hat{n}\, dS = \iint_{S_4} 8y\, dy\, dz$

$$= \int_{y=0}^{3}\int_{z=0}^{3} 8y\, dy\, dz = \cancel{8\times 3 \cdot \frac{9}{2}} = 4\cdot\frac{3^3}{3} = 4\times 9$$

$$= 36$$

on S_5 (curved surface) $\Rightarrow y^2+z^2=9$

$\hat{n} = \frac{1}{3}(y\hat{j} + z\hat{k})$

$\Rightarrow \vec{F}\cdot\hat{n} = \frac{1}{3}(-y^3 + 4xz^3)$.

$\hat{n}\cdot\hat{j} = y/3$.

$$\iint_{S_5} \vec{A}\cdot\hat{n}\,ds = \iint_{S_2} \vec{A}\cdot\hat{n}\,\frac{dx\,dz}{|\hat{n}\cdot\hat{j}|}$$

$$= \iint_{S_2} \frac{-y^3+4xz^3}{y}\,dx\,dz.$$

$$= \iint_{S}\left(-y^2 + \frac{4xz^3}{y}\right)dx\,dz$$

$$= \int_{x=0}^{2}\int_{z=0}^{3}\left\{(z^2-9) + \frac{4xz^3}{\sqrt{9-z^2}}\right\}dx\,dz.$$

$$= \int_{z=0}^{3}\left\{2(z^2-9) + 2\cdot(4)\cdot\frac{z^3}{\sqrt{9-z^2}}\right\}dz$$

$$= 2\left(\frac{3^3}{3} - 9\cdot 3\right) + 8\cdot\int_{z=0}^{3}\frac{z^3}{\sqrt{9-z^2}}\,dz \quad (\text{Put } z = 3\sin\theta)$$

$$= 2(0) + 8\times 27\cdot\frac{2}{3} = \underline{144}.$$

$$\Rightarrow \iint_S \vec{A}\cdot\hat{n}\,ds = 36+144 = \underline{\underline{180}}.$$

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END OF THE EXAMINATION

$\operatorname{div}\vec{A} = 4xy - 2y + 8xz.$

$$\iiint_V \operatorname{div}\vec{A}\, dv = \iint_{S_1} \int_{x=0}^{2} \operatorname{div}\vec{A}\, dx\, dy\, dz = \iint_{S_1} (8y - 4y + 16z)\, dy\, dz.$$

Put $y = r\cos\theta$; $z = r\sin\theta$; $r: 0$ to 3; $0 \le \theta \le \pi/2$

$$= \int_{r=0}^{3}\int_{\theta=0}^{\pi/2} (4r^2\cos\theta + 16r^2\sin\theta)\, dr\, d\theta$$

$$= \frac{3^3}{3} \int_{\theta=0}^{\pi/2} (4\cos\theta + 16\sin\theta)\, d\theta = 9(4+16)$$

$$= \underline{180}$$

$$\therefore \iint_S \vec{A}\cdot\hat{n}\, ds = \iiint_V \operatorname{div}\vec{A}\, dv.$$

ROUGH SPACE

$2\cos\theta\sin\theta$

$$\begin{bmatrix} a & 0 \\ c & b \end{bmatrix}\begin{bmatrix} a & 0 \\ c & b \end{bmatrix} = \Big[4-(2)^2 z$$

$$\frac{dy}{dx} = \frac{1+2\sin y + \cos y}{3+2\sin x + \cos x}$$

$2.(4)y - 4y + 4.4z.$

$3dy + (3\sin x + \cos x)\, dy$

$dx + (2\sin y + \cos y)\, dx$

$\iint(-2y + 3xy^2)$

$= dx + (2\sin y + \cos y)$

$0 = 3C_2 - (9) - 3$

$M\,dx + N\,dy$

$2\sin y + \cos y = v$

$2\sin x + \cos y = u.$

$-(4) + 2x(8)$

$= 2\cos y - \sin y.$

$-4.2 + 8.\frac{4}{2}$ $\frac{\partial M}{\partial y}$

$-8 + 16 = 8.$

$\frac{\partial N}{\partial x} = -(2\cos x - \sin x).$

$3C_2 = 9 + 3$

$C_2 = \frac{dy}{1+2\sin y + \cos y}$

$\frac{dx}{3+2\sin x + \cos x}$

$\frac{2\cos y - \sin y}{2\cos x - \sin x}\frac{dy}{dx} = \frac{dv}{du}.$