A CONSOLIDATED QUESTION PAPER-CUM-ANSWER BOOKLET

IMS™

(INSTITUTE OF MATHEMATICAL SCIENCES)

TEST SERIES (MAIN)-2014

Test Code: FULL TEST P-I(M) IAS / T- 07

MATHEMATICS

by K. VENKANNA

The person with 14 years of Teaching Experience

FULL TEST P-I

Time: Three Hours — *Maximum Marks: 250*

INSTRUCTIONS

1. This question paper-cum-answer booklet has 50 pages and has 33PART/SUBPART questions. Please ensure that the copy of the question paper-cum-answer booklet you have received contains all the questions.
2. Write your Name, Roll Number, Name of the Test Centre and Medium in the appropriate space provided on the right side.
3. A consolidated Question Paper-cum-Answer Booklet, having space below each part/sub part of a question shall be provided to them for writing the answers. Candidates shall be required to attempt answer to the part/sub-part of a question strictly within the pre-defined space. Any attempt outside the pre-defined space shall not be evaluated."
4. Answer must be written in the medium specified in the admission Certificate issued to you, which must be stated clearly on the right side. No marks will be given for the answers written in a medium other than that specified in the Admission Certificate.
5. Candidates should attempt Question Nos. 1 and 5, which are compulsory, and any THREE of the remaining questions selecting at least ONE question from each Section.
6. The number of marks carried by each question is indicated at the end of the question. Assume suitable data if considered necessary and indicate the same clearly.
7. Symbols/notations carry their usual meanings, unless otherwise indicated.
8. All questions carry equal marks.
9. All answers must be written in blue/black ink only. Sketch pen, pencil or ink of any other colour should not be used.
10. All rough work should be done in the space provided and scored out finally.
11. The candidate should respect the instructions given by the invigilator.
12. The question paper-cum-answer booklet must be returned in its entirety to the invigilator before leaving the examination hall. Do not remove any page from this booklet.

READ INSTRUCTIONS ON THE LEFT SIDE OF THIS PAGE CAREFULLY

Name: Nitish.K

Roll No.: 149709

Test Centre: Bangalore

Medium: English

Do not write your Roll Number or Name anywhere else in this Question Paper-cum-Answer Booklet.

I have read all the instructions and shall abide by them.

Signature of the Candidate

I have verified the information filled by the candidate above

Signature of the invigilator

IMPORTANT NOTE:

Whenever a question is being attempted, all its parts/ sub-parts must be attempted contiguously. This means that before moving on to the next question to be attempted, candidates must finish attempting all parts/ sub-parts of the previous question attempted. This is to be strictly followed.

INDEX TABLE

QUESTION	No.	PAGE NO.	MAX. MARKS	MARKS OBTAINED
1	(a)			
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SECTION-A

1. (a) (i) If $A^2 = I$, what are the possible eigen values of A ?
(ii) If this A is 2 by 2, and not I or $-I$, find its trace and determinant.
(iii) If the first row is $(3, -1)$, what is the second row. (10)

(i) $A \cdot A = I \Rightarrow |A| \cdot |A| = 1 \Rightarrow |A|^2 = 1 \Rightarrow |A| = \pm 1$

but $|A|$ = product of eigenvalues $= \lambda_1 \lambda_2 \lambda_3 \dots \lambda_n = \pm 1$

as λ_i is integer $\Rightarrow$ each $\boxed{\lambda_i = \pm 1}$ $\forall i$

$\therefore$ possible eigen values are 1 & -1

(ii) $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$; $A^2 = I$

$\Rightarrow a = -d$ and $a^2 + bc = 1$.

$\Rightarrow A = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$ & $|A| = -a^2 - bc = -1$.

$\begin{cases} \text{Trace } A = a - a = \underline{0}. \\ \det A = -\underline{1} \end{cases}$

iii) $a = 3,\ b = -1 \Rightarrow \underline{d = -a = -3}$.

$a^2 + bc = 1 \Rightarrow 9 - c = 1 \Rightarrow \underline{c = 8}$.

$\therefore$ <u>II row is $(8, -3)$</u>.

$A = \begin{bmatrix} 3 & -1 \\ 8 & -3 \end{bmatrix}$

1. (b) Obtain eigen values and eigen vectors of the differential operator $D : P_2 \to P_2$

$D(a_0 + a_1x + a_2x^2) = a_1 + 2a_2x$, for $a_0, a_1, a_2 \in \mathbb{R}$. **[10]**

$S = \{1, x, x^2\}$ is a basis of P_2

$D(1) = 0$

$D(x) = D(0 + 1\cdot x + 0x^2) = 1$

$D(x^2) = D(0 + 0x + 1\cdot x^2) = 2x$

$$[D] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} = A.$$

for eigen values $\Rightarrow |A - \lambda I| = 0$

$\Rightarrow$ $\underline{\lambda = 0}$ is an eigen value.

eigen vector is $(A - \lambda I)X = 0$

$\Rightarrow AX = 0$ where $X = [x_1 \; x_2 \; x_3]^T$

$\Rightarrow 0x_1 + x_2 + 0x_3 = 0$

$0x_1 + 0x_2 + 2x_3 = 0.$

$\Rightarrow x_2 = 0 ; x_3 = 0$ & $x_1 = 1$ (say)

$\Rightarrow X = [1 \; 0 \; 0]^T$ is eigenvector of A

corresponding to eigenvalue $\underline{\lambda = 0}$.

1. (c) Let ϕ be a function of two variables defined as
$\phi(x, y) = (x^3 + y^3)/(x - y)$, when $x \neq y$
$\phi(x, y) = 0$, when $x = y$
Show that ϕ is discontinuous at the origin, but the first order partial derivatives exist at that point. (10)

$$\phi_x(0,0) = \lim_{h \to 0} \frac{\phi(h,0) - \phi(0,0)}{h} = \lim_{h \to 0} \frac{h^3/h}{h} = \lim_{h \to 0} h$$

$$= 0$$

$$\phi_y(0,0) = \lim_{k \to 0} \frac{\phi(0,k) - \phi(0,0)}{k} = \lim_{k \to 0} \frac{k^3}{-k^2} = \lim_{k \to 0} (-k)$$

$$= 0$$

$\therefore$ both ϕ_x & ϕ_y exists at origin

$\lim\limits_{(x,y) \to (0,0)} \phi(x,y) = ?$; Put $\underline{x - y = mx^3}$. and let $x \to 0$.

$$= \lim_{x \to 0} \frac{x^3 + (x - mx^3)^3}{mx^3}$$

$$= \lim_{x \to 0} \frac{1 + (1 - mx^2)^3}{m} = \frac{1}{m}$$

which is different for different values of m

$\therefore \lim\limits_{(x,y) \to (0,0)} \phi(x,y)$ doesnot exists

$\therefore \phi(x,y)$ is $\underline{\underline{\text{discontinuous}}}$ at origin.

1. (d) Let $E = \{ (x, y) \in \mathbb{R}^2 / 0 < x < y \}$. Then evaluate

$$\iint_E y e^{-(x+y)} dx dy$$ (10)

$$I = \iint_E y e^{-(x+y)} dx\, dy.$$

$$= \int_{y=0}^{\infty} \int_{x=0}^{y} y e^{-y} . \left(e^{-x} dx\right) dy.$$

$$= \int_{y=0}^{\infty} y e^{-y} \left(\frac{e^{-x}}{-1}\right)_{x=0}^{y} dy$$

$$= \int_{y=0}^{\infty} y e^{-y} \left(1 - e^{-y}\right) dy.$$

$$= \int_{y=0}^{\infty} y e^{-y} dy - \int_{y=0}^{\infty} y e^{-2y} dy$$

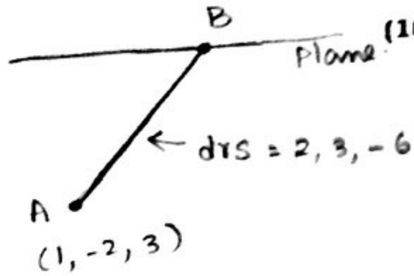
$$= \left[y\left(\frac{e^{-y}}{-1}\right) - \left(\frac{e^{-y}}{1}\right) - \left\{ y\left(\frac{e^{-2y}}{-2}\right) - \left(\frac{e^{-2y}}{4}\right) \right\} \right]_0^{\infty}$$

$$= \left[-y e^{-y} - e^{-y} + \frac{y}{2} e^{-2y} + \frac{e^{-2y}}{4} \right]_0^{\infty}$$

$$= \left[\underline{-(y+1) e^{-y} + \frac{1}{4}(2y+1) e^{-2y}} \right]_0^{\infty}$$

$$= 0 - \left[-(1) + \frac{1}{4}(1) \right] = \frac{3}{4} \, //.$$

1. (e) Find the distance of the point (1, −2, 3) from the plane $x-y+z=5$ measured parallel to the line $\frac{1}{2}x=\frac{1}{3}y=-\frac{1}{6}z$. (10)



equation of the line AB.

$$\frac{x-1}{2}=\frac{y+2}{3}=\frac{z-3}{-6}=r$$

Any point is $(1+2r,\ -2+3r,\ 3-6r)$

if this is B, then it lies on the plane

$$x-y+z=5$$

$$\Rightarrow 1+2r+2-3r+3-6r=5$$

$$-7r=-1 \Rightarrow \boxed{r=1/7}$$

$$\therefore B=\left(\frac{9}{7},\ -\frac{11}{7},\ \frac{15}{7}\right).$$

$$\therefore AB=\sqrt{\left(\frac{9}{7}-1\right)^2+\left(-\frac{11}{7}+2\right)^2+\left(\frac{15}{7}-3\right)^2}$$

$$\boxed{AB=\underline{1}}$$

2. (a) (i) If $A=\begin{bmatrix}4 & 3\\ 1 & 2\end{bmatrix}$, find A^{100} by diagonalizing A.

(ii) Show by direct calculation that AB and BA have the same trace when $A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$ and $B=\begin{bmatrix}q & r\\ s & t\end{bmatrix}$. Deduce that $AB-BA=I$ is impossible (except in infinite dimensions).

(10+5=15)

$$\begin{vmatrix}4-\lambda & 3\\ 1 & 2-\lambda\end{vmatrix}=0 \Rightarrow (\lambda-4)(\lambda-2)-3=0$$

$$\Rightarrow \lambda^2-6\lambda+5=0 \Rightarrow \lambda=5 \text{ \& } \lambda=1.$$

$x_1=[3,1]^T$; $x_2=[-1,1]^T$ are eigen vectors corresponding to eigen values 5 & 1

$$\Rightarrow P^{-1}AP=D=\begin{bmatrix}5 & 0\\ 0 & 1\end{bmatrix}$$

$$P=\begin{bmatrix}3 & -1\\ 1 & 1\end{bmatrix}$$

$$\Rightarrow A=PDP^{-1}$$

$$\Rightarrow A^{100}=PD^{100}P^{-1}$$

$$=\begin{bmatrix}3 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}5^{100} & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\ -1 & 3\end{bmatrix}\frac{1}{4}$$

$$=\frac{1}{4}\begin{bmatrix}3 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}5^{100} & 5^{100}\\ -1 & 3\end{bmatrix}$$

$$\boxed{A^{100}=\frac{1}{4}\begin{bmatrix}3\times5^{100}+1 & 3\times5^{100}-3\\ 5^{100}-1 & 5^{100}+3\end{bmatrix}}$$ A.

$$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} q & r \\ s & t \end{bmatrix} = \begin{bmatrix} aq+bs & ar+bt \\ aq+ds & cr+dt \end{bmatrix}$$

$$BA = \begin{bmatrix} q & r \\ s & t \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} aq+cr & bq+dr \\ as+ct & bs+dt \end{bmatrix}$$

trace $(AB) = aq + bs + cr + dt$

trace $(BA) = aq + cr + bs + dt$

$\Rightarrow$ <u>trace (AB) = trace (BA)</u>

if $AB - BA = I$

$\Rightarrow$ trace $(AB - BA)$ = trace I

$\Rightarrow$ trace (AB) − trace $(BA) = n$ $\quad\because$ trace (AB) = trace (BA)

$\Rightarrow$ <u>$0 = n$</u> which is impossible

$\therefore$ $AB - BA = I$ is not possible.

2. (b) Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ be defined by

$T(x_1, x_2, x_3) = (x_1 - x_2 + 2x_3,\ 2x_1 + x_2,\ -x_1 - 2x_2 + 2x_3)$ **(15)**

Find $R(T)$ and Ker T.

$$\left.\begin{aligned} T(e_1) &= T(1,0,0) = (1,2,-1) \\ T(e_2) &= T(0,1,0) = (-1,1,-2) \end{aligned}\right\} \text{span } R(T)$$

$$T(e_3) = T(0,0,1) = (2,0,2)$$

$$\begin{bmatrix} 1 & 2 & -1 \\ -1 & 1 & -2 \\ 2 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & -3 \\ 0 & -4 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow \dim R(T) = 2$ and

$u_1 = (1, 2, -1)$ forms basis of $R(T)$

$u_2 = (0, 1, -1)$

$$T(x_1, x_2, x_3) = 0 \Rightarrow \begin{aligned} x_1 - x_2 + 2x_3 &= 0 \\ 2x_1 + x_2 \quad &= 0 \\ -x_1 - 2x_2 + 2x_3 &= 0 \end{aligned}$$

$$\begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ -1 & -2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -4 \\ 0 & -3 & 4 \end{bmatrix} \sim \begin{bmatrix} +1 & -1 & 2 \\ 0 & 3 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_1 - x_2 + 2x_3 = 0$$

$$3x_2 - 4x_3 = 0.$$

rank $r = 2$; $n = 3 \Rightarrow \underline{n - r = 1}$.

P.T.O.

$\dim \ker T = 1$.

$x_3 = 3 \Rightarrow x_2 = 4 \Rightarrow x_1 - 4 + 6 = 0$

$\Rightarrow x_1 = -2$.

$\therefore [-2, 4, 3]^T$ forms a basis of $\ker T$.

2. (c) (i) Show that the diagonal entries of a skew symmetric matrix are all zero, but the converse is not true.
(ii) Let $Q' = \{ ax^2 + bx + c \,/\, a \neq 0, a, b, c \in C \}$
Is Q' a complex vector space? Justify your answer.

(5+5=10)

for a skew symmetric matrix $\Rightarrow a_{ij} = -a_{ji}$

$\Rightarrow a_{ii} = -a_{ii} \Rightarrow 2a_{ii} = 0 \Rightarrow a_{ii} = 0 \ \forall i$.

$\therefore a_{11} = 0; \ a_{22} = 0, \ a_{33} = 0, \ldots$ so on.

$\therefore$ diagonal entries of skew symmetric matrix all all zero.

Consider $A = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$; $A^T = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$

clearly $\boxed{A^T \neq -A}$ $\therefore$ A is not skew symmetric even though all diagonal entries are zero.

We know that the space of all polynomials with complex coefficients is a vector space.

i.e $C(x) = \{ a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots\ldots \mid \forall a_i \in \mathbb{C} \}$

We shall show Q' is not a subspace of $C(x)$, thus proving Q' is also not a vector space.

i) $0 = 0 + 0x + 0x^2 \in Q'$

ii) $ax^2 + bx + c \in Q'$ & $px^2 + qx + r \in Q'$

$\Rightarrow (ax^2 + bx + c) + (px^2 + qx + r)$

$= (a+p)x^2 + (b+q)x + (c+r) \in Q'$ as $a+p \neq 0$.

clearly $\underline{\underline{0 \notin Q'}}$

$\therefore$ Q' cannot be a subspace of $C(x)$

2) Q' cannot be a vector space.

2. (d) Let V and W be subspaces of $\mathbf{R}^3$ defined as follows:
$V = \{(a, b, c) / b + 2c = 0\}$, $W = \{(a, b, c)/a + b + c = 0\}$
Find bases and dimensions of V, W, $V \cap W$. Hence prove that
$V + W = \mathbf{R}^3$
(10)

$V = \{(a,b,c) \mid b+2c=0\} = \{(a,-2c,c) \mid a, c \in R\}$

$= \{a(1,0,0) + c(0,-2,1) \mid a, c \in R\}$.

$\therefore u_1 = (1,0,0)$ & $u_2 = (0,-2,1)$ is basis of V as they are linearly independent

$\dim V = 2$.

$W = \{(a,b,c) \mid a+b+c=0\}$

$= \{(-b-c, b, c) \mid b, c \in R\}$

$= \{b(-1,1,0) + c(-1,0,1) \mid b, c \in R\}$

$\Rightarrow w_1 = (-1,1,0)$ & $w_2 = (-1,0,1)$ form basis of W as they are linearly independent

$\dim W = 2$.

$V \cap W = \left\{(a,b,c) \;\middle|\; \begin{matrix} a+b+c=0 \\ b+2c=0 \end{matrix}\right\}$.

$= \{(c,-2c,c) \mid c \in R\}$

$= \{c(1,-2,1) \mid c \in R\}$

$\therefore$ $(1,-2,1)$ is a basis of $V \cap W$

$\dim V \cap W = 1$.

$\dim(V+W) = \dim V + \dim W - \dim(V \cap W)$

$= 2 + 2 - 1 = 3$.

$\therefore$ $\dim(V+W) = 3$ & $V+W \subseteq R^3$

$\Rightarrow$ $\boxed{V+W = R^3}$

3. (a) Discuss the nature of the critical values of $f(x, y) = e^x \sin y$. **(10)**

SECTION-B

5. (a) Find the orothogonal trajectories of cardioids $r = a(1-\cos\theta)$, a being parameter. [10]

$r = a(1-\cos\theta)$.

$\frac{dr}{d\theta} = a(\sin\theta)$. $\Rightarrow a = \operatorname{cosec}\theta \frac{dr}{d\theta}$.

$\Rightarrow r = \operatorname{cosec}\theta (1-\cos\theta) \frac{dr}{d\theta}$.

$r = (\operatorname{cosec}\theta - \cot\theta) \frac{dr}{d\theta}$.

replace $\frac{dr}{d\theta}$ by $-r^2 \frac{d\theta}{dr}$.

$\Rightarrow r = \left(\frac{1}{\sin\theta} - \frac{\cos\theta}{\sin\theta}\right) \times (-r^2) \frac{d\theta}{dr}$

$\Rightarrow -\frac{dr}{r} = \frac{1-\cos\theta}{\sin\theta} d\theta = \frac{2\sin^2\theta/2 \, d\theta}{2\sin\theta/2 \cos\theta/2}$

$\Rightarrow -\frac{dr}{r} = \cancel{\sin} \tan\theta/2 \, d\theta$. ; integrating

$\Rightarrow -\log r = 2\log(\sec\theta/2) - \log c$

$\Rightarrow \frac{c}{r} = \sec^2\theta/2$

$\Rightarrow r = c\cos^2\theta/2 \Rightarrow 2r = c(1+\cos\theta)$

$\Rightarrow \boxed{r = b(1+\cos\theta)}$ is the required orthogonal trajectory

$D^4 + 2D^2 + 1$

5. (b) Solve $(D^2+1)^2 y = 24x\cos x$ given the initial conditions $x=0$, $y=0$, $Dy=0$, $D^2y=0$, $D^3y=12$. **(10)**

$$y^{iv} + 2y'' + y = 24x\cos x.$$

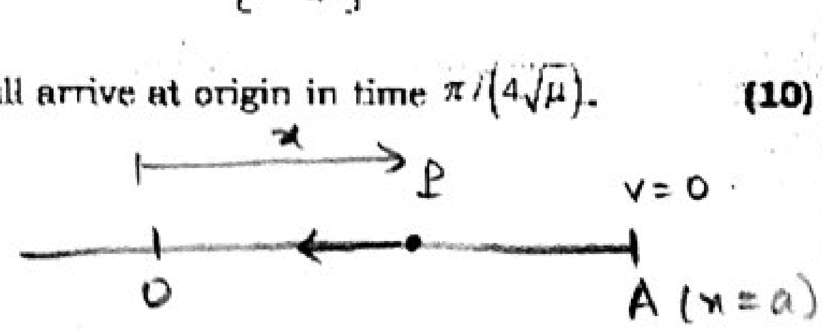
$$s^4 L(y) - s^3 y(0) - s^2 y'(0) - s\,y''(0) - y'''(0)$$
$$+ 2(s^2 L(y) - s\,y(0) - y'(0)) + L(y) = +24\left[\frac{2s^2}{(s^2+1)^2} - \frac{1}{s^2+1}\right]$$

$$(s^4 + 2s^2 + 1) L(y) = 12 + 24\left[\frac{2s^2}{(s^2+1)^2} - \frac{1}{s^2+1}\right].$$

$$L(y) = \frac{12}{(s^2+1)^2} + \frac{48s^2}{(s^2+1)^4} - \frac{24}{(s^2+1)^3}.$$

5. (c) A particle whose mass is m is acted upon by a force $m\mu\left[x+\frac{a^4}{x^3}\right]$ towards origin. If it starts from rest at a distance a, show that it will arrive at origin in time $\pi/(4\sqrt{\mu})$. **(10)**

$$\frac{d^2x}{dt^2} = -\mu\left(x + \frac{a^4}{x^3}\right).$$



$$v^2 = \left(\frac{dx}{dt}\right)^2 = -\mu\left(x^2 - \frac{a^4}{x^2}\right) + A.$$

when $x = a$; $v = 0$ $\Rightarrow$ $A = 0$.

$$\left(\frac{dx}{dt}\right)^2 = \mu\left(\frac{a^4 - x^4}{x^2}\right) \Rightarrow \frac{dx}{dt} = -\sqrt{\mu}\,\frac{\sqrt{a^4 - x^4}}{x}.$$

P.T.O.

$$\int dt = +\frac{1}{2\sqrt{\mu}}\int_{x=a}^{0}\frac{-2x\,dx}{\sqrt{a^4-x^4}} = \cancel{\frac{1}{2\sqrt{\mu}}\cdot\left[2\sqrt{a^4-x^2}\right]_{x=a}^{0}}$$

$$= \cancel{\frac{1}{2\sqrt{\mu}}\left[a^2 - \right.} \Rightarrow t = -\frac{1}{\sqrt{\mu}}\int_{x=a}^{0}\frac{x\,dx}{\sqrt{a^4-x^4}}$$

Put $x^2 = a^2\sin\theta \Rightarrow 2x\,dx = a^2\cos\theta\,d\theta$.

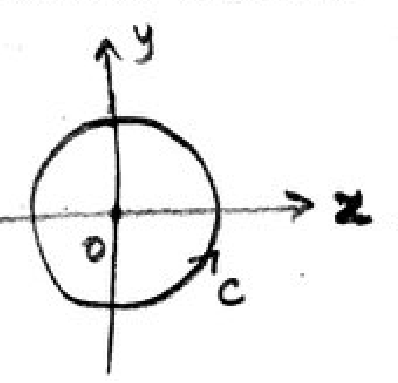
$$t = \frac{1}{\sqrt{\mu}}\int_0^{\pi/2}\frac{a^2}{2}\frac{\cos\theta\,d\theta}{a^2\cdot\cos\theta} = \frac{1}{2\sqrt{\mu}}\int_0^{\pi/2} d\theta = \frac{1}{2\sqrt{\mu}}\frac{\pi}{2}$$

$$\boxed{t = \frac{\pi}{4\sqrt{\mu}}}$$

5\. (d) A solid frustum of a paraboloid of revolution of height h and latus rectum $4a$, rests with its vertex on the vertex of a paraboloid of revolution, whose latus rectum is $4b$, show that the equilibrium is stable if $h < \frac{3ab}{a+b}$. **(10)**

5. (e) Find the work done in moving a particle once around a circle C in the xy-plane, if the circle has centre at the origin and radius 2 and if the force field F is given by

$F = (2x - y + 2z)\mathbf{i} + (x + y - z)\mathbf{j} + (3x - 2y - 5z)\mathbf{k}$. **(10)**



xy-plane $\Rightarrow z = 0,\ dz = 0$.

$\vec{F} = (2x - y)\hat{i} + (x + y)\hat{j} + (3x - 2y)\hat{k}$

$d\vec{r} = dx\,\hat{i} + dy\,\hat{j}$

$$\text{Work done} = \int \vec{F} \cdot dr$$

$$= \int_{C} (2x - y)\,dx + (x + y)\,dy.$$

$x = 2\cos\theta;\ y = 2\sin\theta$

$dx = -2\sin\theta\,d\theta;\ dy = 2\cos\theta\,d\theta$

$$= \int_{\theta=0}^{2\pi} \{(4\cos\theta - 2\sin\theta)(-2\sin\theta) + 2(\cos\theta + \sin\theta)\, 2\cos\theta\}\, d\theta$$

$$= \int_{\theta=0}^{2\pi} (-8\cos\theta\sin\theta + 4\sin^2\theta + 4\cos^2\theta + 4\cos\theta\sin\theta)\, d\theta.$$

$$= \int_{\theta=0}^{2\pi} (4 - 4\cos\theta\sin\theta)\, d\theta.$$

$$= \underline{\underline{8\pi}}.$$

6. (a) Solve $(1 - x^2y^2)\, dx = ydx + xdy$. **(05)**

$$dx = \frac{d(xy)}{1-(xy)^2} = \frac{dt}{1-t^2} \quad \text{where } t = xy$$

integrating

$$x = \int \frac{dt}{1-t^2} = \frac{1}{2}\log\left(\frac{1+t}{1-t}\right) + c$$

$$\Rightarrow \boxed{x = \frac{1}{2}\log\left(\frac{1+xy}{1-xy}\right) + c}$$

6. (b) Solve the differential equation $(px^2 + y^2)(px + y) = (p + 1)^2$ by reducing it to Clairaut's form and find its singular solution. **(15)**

Let $x + y = u$; $xy = v$.

$\Rightarrow dx + dy = du$ & $x\,dy + y\,dx = dv$

$$\frac{x\,dy + y\,dx}{dx + dy} = \frac{dv}{du} = P \Rightarrow \frac{px + y}{1 + p} = P$$

$$\Rightarrow p = \frac{P - y}{x - P} \quad \text{— (1)}$$

using (1)

$$\therefore (px^2 + y^2)\left(\frac{px + y}{p + 1}\right) = p + 1$$

$$\Rightarrow \frac{(P - y)x^2 + y^2(x - P)}{x - P} \cdot P = \frac{x - y}{x - P}.$$

$$\Rightarrow [P(x^2 - y^2) - xy(x - y)]P = x - y$$

$$\Rightarrow [P(x + y) - xy]P = 1$$

$$\Rightarrow [Pu - v] = 1/P \Rightarrow \boxed{v = Pu - \frac{1}{P}}$$

This is in Clairaut's form

$\therefore v = cu - \frac{1}{c}$ is the solution.

$$\boxed{xy = c(x + y) - \frac{1}{c}.}$$

$$\Rightarrow cxy = c^2(x + y) - 1.$$

$$\Rightarrow c^2(x + y) - cxy - 1 = 0.$$

$\therefore$ c-discriminate equation & $B^2 - 4AC = 0$

P.T.O.

$xy^2 + 4(x+y) = 0 \Rightarrow \underline{x^2y^2 + 4x + 4y = 0}$.

$\Rightarrow 2xy^2 + 2x^2yp + 4 + 4p = 0 \Rightarrow xy^2 + x^2yp + 2 + 2p = 0$

$\Rightarrow p(x^2y + 2) = -2 - xy^2 \Rightarrow p = \dfrac{-2-xy^2}{x^2y+2}$.

This satifies given differential equation

$\underline{x^2y^2 + 4(x+y) = 0}$ is the required singular Solution.

5. (c) Solve $(x+2)y'' - (2x+5)y' + 2y = (x+1)e^x$. (15)

$$y'' - \left(\frac{2x+5}{x+2}\right)y' + \frac{2}{x+2}y = \left(\frac{x+1}{x+2}\right)e^x$$

$$y'' + Py' + Qy = R.$$

$$\Rightarrow P = -\left(\frac{2x+5}{x+2}\right); \quad Q = \frac{2}{x+2}; \quad R = \left(\frac{x+1}{x+2}\right)e^x.$$

clearly $2^2 + 2P + Q = 0 \Rightarrow \underline{u = e^{2x}}$ is part of C.F.

Let $y = uv$ be the complete solution

where $\dfrac{d^2v}{dx^2} + \left(P + \dfrac{2}{u}\dfrac{du}{dx}\right)\dfrac{dv}{dx} = \dfrac{R}{u}$.

$$\Rightarrow \frac{d^2v}{dx^2} + \left(-\left(\frac{2x+5}{x+2}\right) + \frac{2}{e^{2x}} \cdot 2e^{2x}\right)\frac{dv}{dx} = \left(\frac{x+1}{x+2}\right)e^{-x}.$$

Let $\frac{dv}{dx} = q$.

$$\frac{dq}{dx} + \left(\frac{2x+3}{x+2}\right) q = \left(\frac{x+1}{x+2}\right) e^{-x}$$

$$\int P\,dx = \int \frac{2x+3}{x+2}\,dx = \int 2 - \frac{1}{x+2}\,dx = 2x - \log(x+2)$$

$$I.F = e^{\int P\,dx} = e^{2x} \cdot \frac{1}{x+2}.$$

$$\therefore \quad q \cdot \frac{e^{2x}}{x+2} = \int \left(\frac{x+1}{x+2}\right) e^{-x} \cdot \frac{e^{2x}}{x+2}\,dx$$

$$= \int \frac{x+2-1}{(x+2)^2} e^{x}\,dx.$$

$$= \int \left[\frac{1}{x+2} - \frac{1}{(x+2)^2}\right] e^{x}\,dx.$$

$$= \frac{1}{x+2}(e^{x}) + \cancel{\int e^{x} \frac{1}{(x+2)^2}\,dx} - \cancel{\int \frac{e^{x}\,dx}{(x+2)^2}}.$$

$$\Rightarrow \quad q \frac{e^{2x}}{x+2} = \frac{e^{x}}{x+2} + c_1.$$

$$\Rightarrow \quad q = \frac{dv}{dx} = e^{-x} + c_1(x+2)e^{-2x}.$$

$$\Rightarrow \quad v = \frac{e^{-x}}{(-1)} + c_1\left[(x+2)\left(\frac{e^{-2x}}{-2}\right) - (1)\left(\frac{e^{-2x}}{4}\right)\right] + c_2.$$

$$\Rightarrow v = -e^{-x} - \frac{c_1}{4}\left[2(x+2)+1\right]e^{-2x} + c_2.$$

$$v = -e^{-x} - \frac{c_1}{4}(2x+5)e^{-2x} + c_2 .$$

$$\therefore y = uv$$

$$\boxed{y = -e^{x} - \frac{c_1}{4}(2x+5) + c_2 e^{2x}}$$

6. (d) By using Laplace transform, solve $(D^2+9)y = \cos 2t$ with $y(0)=1$, $y(\pi/2)=-1$. **(15)**

$y'' + 9y = \cos 2t$; Taking Laplace Tranforms

~~[illegible]~~

$$s^2 L(y) - s\,y(0) - y'(0) + 9L(y) = \frac{s}{s^2+4} .$$

$y(0)=1$; let $y'(0) = A$.

$$\Rightarrow (s^2+9)L(y) - s - A = \frac{s}{s^2+4}$$

$$\Rightarrow L(y) = \frac{s}{s^2+9} + \frac{A}{s^2+9} + \frac{s}{(s^2+9)(s^2+4)}$$

$$= \frac{s}{s^2+9} + \frac{A}{s^2+9} + \frac{1}{5}\left[\frac{s}{s^2+4} - \frac{s}{s^2+9}\right]$$

$$y = \cos 3t + A\frac{\sin 3t}{3} + \frac{1}{5}\left[\cos 2t - \cos 3t\right]$$

$$y = \frac{4}{5}\cos 3t + \frac{1}{5}\cos 2t + \frac{A}{3}\sin 3t .$$

$\Rightarrow t = \pi/2 ;\quad y = -1 .$

$$-1 = \frac{4}{5}(0) + \frac{1}{5}(-1) + \frac{A}{3}(-1)$$

$$\frac{A}{3} = \frac{4}{5} \quad \Rightarrow \quad \boxed{A = \frac{12}{5}}$$

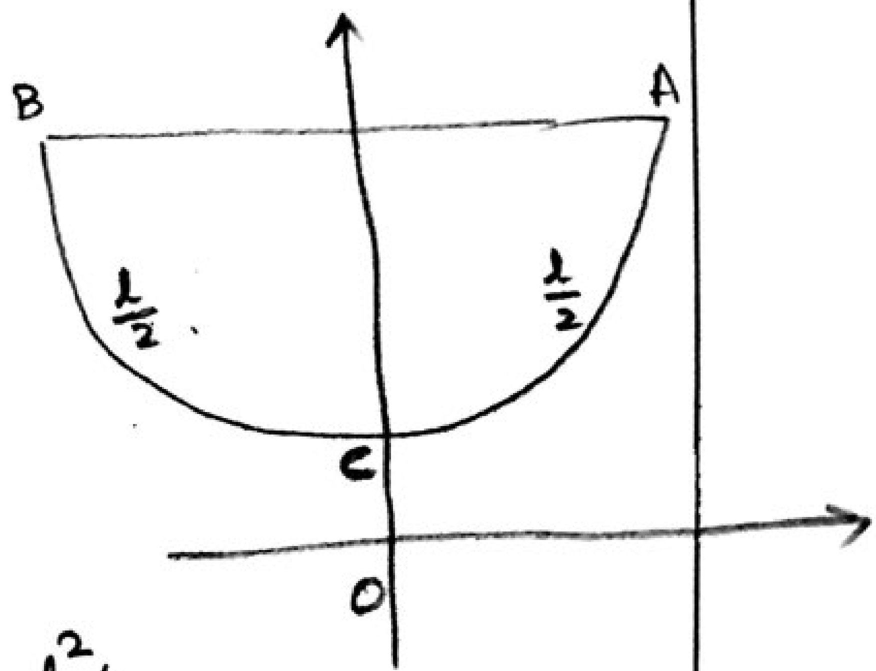
$$\therefore \boxed{y(t) = \frac{4}{5}\cos 3t + \frac{1}{5}\cos 2t + \frac{4}{5}\sin 3t}$$

7. (a) A uniform chain of length l, is to be suspended from two points A and B, in the same horizontal line so that either terminal tension is n times that at the lowest point. Show that the span AB must be $\frac{l}{\sqrt{n^2-1}}\log\left\{n+\sqrt{n^2-1}\right\}$. (16)

$T_A = n\omega c$

$T_A = \omega y_A$

$\Rightarrow \boxed{y_A = nc}$



$y^2 = c^2 + s^2$

$\Rightarrow y_A^2 = c^2 + s_A^2 \Rightarrow n^2c^2 = c^2 + l^2/4$.

$(n^2-1)c^2 = l^2/4 \Rightarrow \boxed{c = \frac{l}{2\sqrt{n^2-1}}}$

$y = c\sec\psi \Rightarrow y_A = c\sec\psi_A$

$\Rightarrow nc = c\sec\psi_A \Rightarrow \boxed{\sec\psi_A = n}$

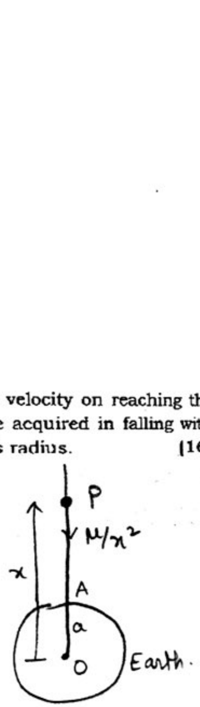
span $AB = 2x_A = 2c\log(\sec\psi_A + \tan\psi_A)$

$= \frac{l}{\sqrt{n^2-1}}\log\left(n + \sqrt{\sec^2\psi_A - 1}\right)$

$$\boxed{AB = \frac{l}{\sqrt{n^2-1}}\log\left(n+\sqrt{n^2-1}\right)}$$

7. (b) A particle falls towards the earth from infinity; show that its velocity on reaching the surface of the earth is the same as that which it would have acquired in falling with constant acceleration g through a distance equal to the earth's radius. [16]

Let P be the position of the particle at any time t.



$$\frac{d^2x}{dt^2} = -\frac{\mu}{x^2}$$

on the surface of earth, A

$$x = a; \quad \frac{d^2x}{dt^2} = -g \Rightarrow -g = -\frac{\mu}{a^2}$$

$$\Rightarrow \mu = a^2 g.$$

$$\Rightarrow \frac{d^2x}{dt^2} = -\frac{a^2 g}{x^2}$$; multiplying both sides by $2\frac{dx}{dt}$ and integrating

$\Rightarrow \quad v^2 = \left(\frac{dx}{dt}\right)^2 = \frac{2a^2 g}{x} + B$

but $x \to \infty$; $v = 0 \Rightarrow B = 0$

$\Rightarrow \cancel{\frac{dx}{dt} = -\sqrt{\frac{2a^2 g}{x}}} \quad \Rightarrow \cancel{dt = -\frac{1}{\sqrt{2a^2 g}}\sqrt{x}\, dx}$

$\cancel{t = -\frac{1}{\sqrt{2a^2 g}} \int x^{1/2}\, dx}$.

$\Rightarrow \boxed{v^2 = \frac{2a^2 g}{x}}$ gives velocity at any distance x from centre of earth O.

on the surface of earth $\Rightarrow x = a$.

$\Rightarrow \boxed{v^2 = 2ag}$ — ①

velocity acquired in falling with constant acceleration g through ~~at~~ distance 'a'

$v_1^2 = u^2 + 2as$; $a = +g$; $u = 0$

$s = a$

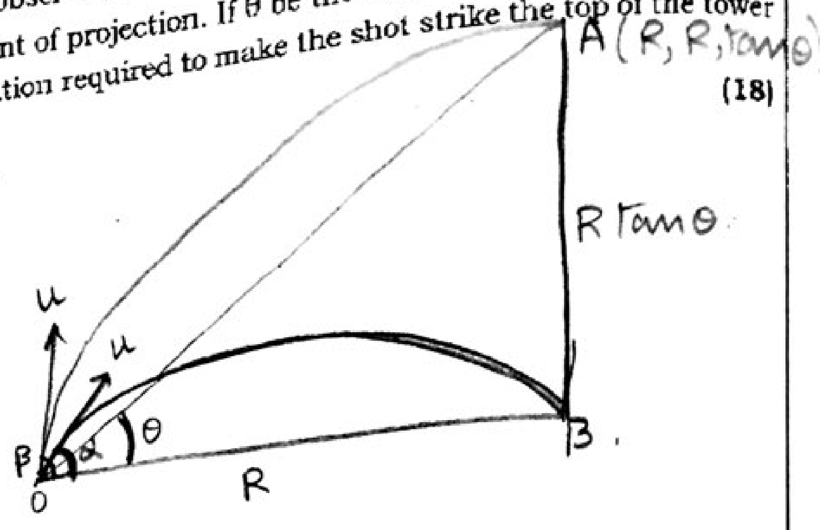
$v_1^2 = 0 + 2ga$.

$\Rightarrow \boxed{v_1^2 = 2ag}$ — ②

from ① & ② $\Rightarrow v^2 = v_1^2$

$\Rightarrow \boxed{v = v_1}$

7. (c) A shot fired at an elevation α is observed to strike the foot of a tower which rises above a horizontal plane through the point of projection. If θ be the angle subtended by the tower at this point, show that the elevation required to make the shot strike the top of the tower is $\frac{1}{2}\left[\theta + \sin^{-1}(\sin\theta + \sin 2\alpha \cos\theta)\right]$. **(18)**



The point A (R, R tanθ) lies on the trajectory

$$y = x\tan\beta - \frac{1}{2} g \frac{x^2}{u^2\cos^2\beta}.$$

$$R\tan\theta = R\tan\beta - \frac{gR^2}{2u^2}\sec^2\beta.$$

but $R = \frac{u^2 \sin 2\alpha}{g}$.

$$\tan\theta = \tan\beta - \frac{g}{2u^2} R\sec^2\beta$$

$$\tan\theta = \tan\beta - \frac{1}{2}\frac{\sin 2\alpha}{\cos^2\beta}.$$

$$\frac{\sin\theta}{\cos\theta} = \frac{\sin\beta}{\cos\beta} - \frac{1}{2}\frac{\sin 2\alpha}{\cos^2\beta}.$$

$$2\sin\theta\cos^2\beta = 2\sin\beta\cos\beta\cos\theta - \sin 2\alpha\cos\theta$$

$\Rightarrow$ ~~$\sin 2\alpha \cos\theta = 2\cos\beta[\sin\beta\cos\theta$~~

$\Rightarrow$ $2\sin\theta\cos^2\beta = \sin 2\beta\cos\theta - \sin 2\alpha\cos\theta$

$\Rightarrow$ $\sin\theta(1+\cos 2\beta) = \sin 2\beta\cos\theta - \sin 2\alpha\cos\theta.$

$\Rightarrow$ $\sin\theta + \sin 2\alpha\cos\theta = \sin 2\beta\cos\theta - \sin\theta\cos 2\beta$

$\Rightarrow \sin\theta + \sin 2\alpha \cos\theta = \sin(2\beta - \theta)$

$\Rightarrow 2\beta - \theta = \sin^{-1}(\sin\theta + \sin 2\alpha \cos\theta)$

$$\boxed{\beta = \frac{1}{2}\left[\theta + \sin^{-1}\left(\sin\theta + \sin 2\alpha \cos\theta\right)\right]}$$

8\. (a) A particle moves along the curve $x = e^{t}$, $y = 2\cos 3t$, $z = 2\sin 3t$. Determine the velocity and acceleration at any time t and their magnitudes at $t = 0$. **[08]**